Latent Curve Modeling

Lecture Notes

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CenterStat.org

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Symbol Definitions

N	Total sample size
i	An individual sample unit, where $i = 1, 2, 3, \ldots N$
n_i	Number of observed variable values for case i; e.g., $n_i = 4$ if case i has observed data for four of five variables in the model)
k	The number of unique population means, variances, and covariances for the set of observed variables in a model
t	The number of parameters to be estimated for a model
p	The number of endogenous variables in a model
q	The number of exogenous variables in a model
r	The number of latent variables (factors) in a model
L	Likelihood of a model (used in reference to maximum likelihood estimation)
ℓ	Log-likelihood of a model (used in reference to maximum likelihood estimation)
T	Test statistic comparing a hypothesized structural equation model to the saturated model (i.e., test of overall model fit), chi-square distributed with $k - t$ degrees of freedom under certain assumptions and conditions; may be subscripted to specify a particular model (e.g., by h or b to specify hypothesized or baseline model, respectively).
y	A single endogenous observed variable (a.k.a. an outcome or criterion variable)
x	A single exogenous observed variable (a.k.a. a predictor, covariate, or independent variable)
z	A time-varying covariate, or TVC.
η	Eta; a single latent variable
μ	Mu; population mean for a single observed variable
σ^2	Sigma-squared; population variance for a single observed variable
$E(\cdot)$	Expected value operator; returns the long-run average or population mean; e.g., $E(y) = \mu$
$VAR(\cdot)$	Variance operator; returns the population variance; e.g., $VAR(y) = \sigma^2$
$\bar{y}$, $\bar{x}$	Sample mean for y and x, respectively.

s^2 Sample variance for a single observed variable (may be subscripted by x or y for specificity)

$\mathbf{y}$ A p-length vector of endogenous observed variables (a.k.a. outcomes or criterion variables)

$\mathbf{x}$ A q-length vector of exogenous observed variables (a.k.a. predictors, covariates, or independent variables)

$\mathbf{z}$ Combined $(p + q)$-length vector of $\mathbf{y}$ and $\mathbf{x}$ ($\mathbf{y}$ stacked on top of $\mathbf{x}$ in a single vector); also sometimes used to denote a time-varying covariate in multivariate growth models

$\boldsymbol{\eta}$ Eta; r-length vector of latent variables (latent factors)

$\boldsymbol{\theta}$ Theta; a t-length vector consisting of all the parameters in a given structural equation model

$\hat{\boldsymbol{\theta}}$ Theta-hat; a t-length vector consisting of all the sample estimates for the parameters of a given structural equation model

$\boldsymbol{\mu}$ Mu; population mean vector for a set of observed variables (may be subscripted by $\mathbf{x}$ or $\mathbf{y}$ to make specific to exogenous or endogenous variables)

$\boldsymbol{\Sigma}$ Sigma; population covariance matrix for a set of observed variables (may be subscripted by $\mathbf{xx}$, $\mathbf{yy}$, or $\mathbf{xy}$ to specifically reference covariances of exogenous variables, endogenous variables, or exogenous with endogenous variables, respectively)

$\boldsymbol{\mu}(\boldsymbol{\theta})$ The model-implied population mean vector for a set of observed variables, implied by a structural equation model with parameters $\boldsymbol{\theta}$ (where $\boldsymbol{\mu}$ may be subscripted by $\mathbf{x}$ or $\mathbf{y}$ to make specific to exogenous or endogenous variables)

$\boldsymbol{\Sigma}(\boldsymbol{\theta})$ The model-implied population covariance for a set of observed variables, implied by a structural equation model with parameters $\boldsymbol{\theta}$ (where $\boldsymbol{\Sigma}$ may be subscripted by $\mathbf{xx}$, $\mathbf{yy}$, or $\mathbf{xy}$ to specifically reference model-implied covariances of exogenous variables, endogenous variables, or exogenous with endogenous variables, respectively)

$\mathbf{m}$ The sample mean vector for a set of observed variables (may be subscripted by $\mathbf{x}$ or $\mathbf{y}$ to make specific to exogenous or endogenous variables)

$\mathbf{S}$ The sample covariance matrix for a set of observed variables (may be subscripted by $\mathbf{xx}$, $\mathbf{yy}$, or $\mathbf{xy}$ to specifically reference covariances of exogenous variables, endogenous variables, or exogenous with endogenous variables, respectively)

$\boldsymbol{\mu}\left(\hat{\boldsymbol{\theta}}\right)$ The sample estimated model-implied mean vector for a set of observed variables implied by a structural equation model with parameters $\boldsymbol{\theta}$ (where $\boldsymbol{\mu}$ may be subscripted by $\mathbf{x}$ or $\mathbf{y}$ to make specific to exogenous or endogenous variables)

$\boldsymbol{\Sigma}\left(\hat{\boldsymbol{\theta}}\right)$ The sample estimated model-implied population covariance for a set of observed variables implied by a structural equation model with parameters $\boldsymbol{\theta}$ (where $\boldsymbol{\Sigma}$ may be subscripted by $\mathbf{xx}$, $\mathbf{yy}$, or $\mathbf{xy}$ to specifically reference model-implied covariances of exogenous variables, endogenous variables, or exogenous with endogenous variables, respectively)

$\boldsymbol{\alpha}$ Alpha; vector of intercepts for endogenous latent variables (in full SEM with latent variables) or endogenous observed variables (in path analysis / simultaneous equation model)

$\boldsymbol{\Gamma}$ Gamma; matrix of regression slopes capturing causal effects of observed exogenous variables on latent variables (in full SEM) or endogenous observed variables (in path analysis / simultaneous equation model)

$\mathbf{B}$ Beta; matrix of regression slopes capturing causal effects of latent variables on other latent variables (in full SEM with latent variables) or of endogenous observed variables on other endogenous observed variables (in path analysis / simultaneous equation model)

$\boldsymbol{\zeta}$ Zeta; vector of residuals (or disturbances) for latent variables (in full SEM with latent variables) or endogenous observed variables (in path analysis / simultaneous equation model)

$\boldsymbol{\Psi}$ Psi; matrix of residual variances and covariances for latent variables (in full SEM with latent variables) or endogenous observed variables (in path analysis / simultaneous equation model); i.e., $VAR\left(\zeta_i\right) = \boldsymbol{\Psi}$

$\boldsymbol{\nu}$ Nu; vector of residuals for observed indicators of latent factors in full SEM with latent variables

$\boldsymbol{\kappa}$ Kappa; a regression coefficient associated with a time-varying covariate effect

$\boldsymbol{\Lambda}$ Lambda; matrix of factor loadings (regression slopes) relating latent factors to observed indicator variables; individual elements denoted λ

$\boldsymbol{\varepsilon}$ Epsilon; vector of residuals for observed indicator variables after accounting for the influence of latent factors

$\boldsymbol{\Theta}$ Theta; matrix of residual variances and covariances for observed indicator variables after accounting for the influence of the latent factors; i.e., $VAR(\boldsymbol{\varepsilon}_i) = \boldsymbol{\Theta}$

h^2 Communality, the proportion of variance in an observed variable that is accounted for by the latent variables in the model (equivalent to a multiple R^2 from a regression of the indicator on all related latent factors); used in factor analysis to gauge the quality of indictor variables.

Chapter 1
Introduction

1.1 Overview and Structure of the Workshop 1-3
1.2 Two Time Point Designs and Raw Change Scores 1-10

1.1 Overview and Structure of the Workshop

Objectives

- Define what we mean by longitudinal data and review several common designs used in practice
- Describe different types of longitudinal data to which the latent curve model is often applied
- Provide a conceptual framing for the research questions we typically seek to address when collecting and analyzing longitudinal data
  - Focus on inter-individual differences in intra-individual change
- Describe goals and structure for this workshop

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What is "Longitudinal" Data?

- Baltes & Nesselroade (1979):
  - "*The one sine qua non of longitudinal research is that the entity under investigation is observed repeatedly as it exists and evolves over time*"
- We adopt this minimalist definition with one caveat:
  - at least a *subset* of entities under investigation are observed repeatedly
  - allows for missing data and single observation data
- It is helpful to consider several common types of longitudinal designs

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Baltes, P.B., & Nesselroade, J.R. (Eds.). (1979). *Longitudinal Research in the Study of Behavior and Development*. Academic Press.

Cohort-by-Age Design Structure

- A cohort-by-age table is helpful to organize different designs
  - **rows** are *birth cohorts* (or *birth years*) and **columns** are *chronological ages*
  - **cells** are the year in which a child from a given cohort was a given age
  - for example, a child in the 1990 cohort was 16 years of age in 2006

	Chronological Age						
Cohort	10	12	14	16	18	20	22
1986	1996	1998	2000	2002	2004	2006	2008
1988	1998	2000	2002	2004	2006	2008	2010
1990	2000	2002	2004	2006	2008	2010	2012
1992	2002	2004	2006	2008	2010	2012	2014
1994	2004	2006	2008	2010	2012	2014	2016
1996	2006	2008	2010	2012	2014	2016	2018

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Single Time-point, Single Age Design

- The simplest design consists of a single assessment of a single age
  - with $1.50 to conduct your research, you could only assess kids in a single class who were all 12 in 2008
- Confounds individual change, age-related change, and cohort differences

	Chronological Age						
Cohort	10	12	14	16	18	20	22
1986	1996	1998	2000	2002	2004	2006	2008
1988	1998	2000	2002	2004	2006	2008	2010
1990	2000	2002	2004	2006	2008	2010	2012
1992	2002	2004	2006	2008	2010	2012	2014
1994	2004	2006	2008	2010	2012	2014	2016
1996	2006	2008	2010	2012	2014	2016	2018

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Cross-Sectional Design

- A more common design is a single assessment collected in 2008, at which multiple ages are assessed as a ***cross-section*** of ages 12 to 22
- This is strictly a *between-person* design as each individual provides just one assessment at their given age in 2008

	Chronological Age						
Cohort	10	12	14	16	18	20	22
1986	1996	1998	2000	2002	2004	2006	**2008**
1988	1998	2000	2002	2004	2006	**2008**	2010
1990	2000	2002	2004	2006	**2008**	2010	2012
1992	2002	2004	2006	**2008**	2010	2012	2014
1994	2004	2006	**2008**	2010	2012	2014	2016
1996	2006	**2008**	2010	2012	2014	2016	2018

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Single Cohort Longitudinal Design

- Instead of holding age constant by changing cohorts over assessment year, can hold cohort constant and change age
  - individual observations are assessed repeatedly over time
- For example, following the 1994 cohort starting in 2004 provides a longitudinal design from age 10 to 22:

	Chronological Age						
Cohort	10	12	14	16	18	20	22
1986	1996	1998	2000	2002	2004	2006	2008
1988	1998	2000	2002	2004	2006	2008	2010
1990	2000	2002	2004	2006	2008	2010	2012
1992	2002	2004	2006	2008	2010	2012	2014
1994	**2004**	**2006**	**2008**	**2010**	**2012**	**2014**	**2016**
1996	2006	2008	2010	2012	2014	2016	2018

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Multiple Cohort Longitudinal Design

- Instead of following a single cohort, could follow multiple cohorts
- Spans seven years of age, but no child assessed more than 3 times
  - thus often called an *accelerated longitudinal design*
  - allows for specific tests of **cohort effects**

	Chronological Age						
Cohort	10	12	14	16	18	20	22
1986	1996	1998	2000	2002	2004	2006	2008
1988	1998	2000	2002	2004	2006	2008	2010
1990	2000	2002	2004	2006	2008	2010	2012
1992	2002	2004	2006	2008	2010	2012	2014
1994	2004	2006	2008	2010	2012	2014	2016
1996	2006	2008	2010	2012	2014	2016	2018

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Multiple Cohort Longitudinal Design

- Multiple cohorts introduce data that are *missing by design*
- For example, 1994 cohort are "missing" ages 16 to 22
  - *by design*, because cohort determines which ages observed for person
  - The multiple cohort design is commonly encountered in practice

	Chronological Age						
Cohort	10	12	14	16	18	20	22
1986	1996	1998	2000	2002	2004	2006	2008
1988	1998	2000	2002	2004	2006	2008	2010
1990	2000	2002	2004	2006	2008	2010	2012
1992	2002	2004	2006	2008	2010	2012	2014
1994	2004	2006	2008	2010	2012	2014	2016
1996	2006	2008	2010	2012	2014	2016	2018

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Panel Data

- Single- and multiple-cohort designs often collectively referred to as **panel data**
  - Large number of individuals (e.g., 100+)
  - Modest number of time points (e.g., 3 to 10)
  - Intervals between time points typically chosen to observe long-term change
  - Intervals are often of equal length (e.g., 1 calendar year)
  - Often observational in nature
- The latent curve model explored in this workshop is particularly well suited to the analysis of panel data

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Other Longitudinal Designs

- The LCM is not, however, limited to panel data
- Works well for any design in which a modest number of repeated measures are obtained on a relatively large number of sample units
  - Time points need not be widely or evenly spaced (e.g., could be motivated by concern for short-term changes over days or weeks)
  - Time points need not be the same for all (e.g., accelerated longitudinal design)
  - Sample units need not be people (e.g., counties, precincts, etc.)
  - Design need not be observational (e.g., RCT with outcome of intervention observed at multiple points before/during/after treatment)
- With many time points may need to move towards intensive longitudinal data analytic approaches such as the multilevel model

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Goals of Longitudinal Data Analysis

- There are many potential reasons to collect longitudinal data
  - establishing temporal order of cause relative to effect, reducing recall bias, distinguishing age from cohort effects, studying measurement over time, etc.
- But primary motivation is often to observe and predict individual variability in stability and change over time
- Baltes & Nesselroade (1979) describe this nicely as the examination of ***inter-individual differences*** in ***intra-individual change***
  1. can characterize the pattern of change ***within*** each individual
  2. can examine differences in individual-change patterns ***across*** individuals
- Latent curve models provide a powerful framework for examining *between-person* differences in *within-person* change

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Organization For The Workshop

- Begin with brief review of change in two time point designs
- Introduce the univariate latent curve model (LCM) and estimate nonlinear trajectories and person-level predictors
- Expand LCMs to two constructs over time
- Examine extensions of the LCM including non-normality, statistical power, multiple group models, and a survey of advanced topics
- We assume prior exposure to the structural equation model (SEM), but see *centerstat.org* for access to a free three-day workshop that describes the foundations of the SEM
  - Also a short pre-recorded review available with this workshop

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Free SEM workshop: https://centerstat.org/structural-equation-modeling/

Free review of matrix algebra (a.k.a. linear algebra): https://centerstat.org/matrix-review/

Summary

- We adopt a minimalist definition of longitudinal data: a subset of entities observed repeatedly over time
- We are concerned particularly with longitudinal data collected over a modest number of time points for a large number of people
  - Definitions of "modest" and "large" intentionally vague
  - Classic example is panel data
- Possibly the biggest advantage of longitudinal data is the ability to study between-person differences in within-person change
  - inter-individual differences in intra-individual change
- LCM provides a powerful framework for examining this

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1.2 Two Time Point Designs and Raw Change Scores

Objectives

- Describe traditional analysis of two-time point data given this is the cornerstone of more advanced models
- Focus on analysis of raw-score change since latent curve model can be viewed as extending upon this approach
- Highlight strengths and potential limitations
  - Motivate move to LCM

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Raw-Score Change

- Let y_1 and y_2 denote measurements of the same outcome variable at times 1 and 2
  - e.g., mother's report of child antisocial behavior at two assessments
- A simple and intuitive way to infer change on y is to take the difference between these two observations

$$d_i = y_{2i} - y_{1i}$$

- This difference between raw scores lies at the foundation of many classical techniques for longitudinal data analysis
  - paired samples t-test, repeated measures ANOVA and MANOVA

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Analysis of Raw-Score Change

- This raw change score can be evaluated in many different ways
- For example, to evaluate potential change over time, we might test the null hypothesis that the mean difference is zero

$$H_0 : \mu_d = 0$$

- Here, $\bar{d}$ represents the sample mean of the individual difference scores
- One-sample t-test of $\bar{d}$ evaluates null hypothesis of "no change"
  - if *t*-test is "significant", then reject null that no change occurred
  - equivalent to *dependent samples t-test* of y_1 and y_2 we all learned in intro stats

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Analysis of Raw-Score Change

- Simple t-test can be extended in a variety of logical ways
  - repeated measures ANOVA for more than two time points
  - inclusion of between-subjects factors (e.g., treatment versus control)
  - using raw change score as DV in multiple regression model
- But limited by problem of *unreliability*
- Lord (1956) noted "*The difference between two fallible measures is frequently much more fallible than either*"
  - interested in *true-score change* but fit model to *raw-score change*
  - inferences about true-score change may be biased and under-powered when analyzing unreliable raw-score change scores

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This is the key issue at hand. Under many conditions commonly encountered in applied research, the difference score can have even *greater* unreliability than either of the time-specific measures on which it is based.

Unreliability of Difference Score

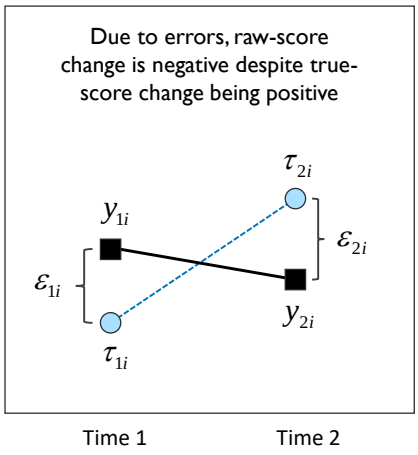
- Can decompose repeated measures into true scores plus error:

$$y_{1i} = \tau_{1i} + \varepsilon_{1i}$$
$$y_{2i} = \tau_{2i} + \varepsilon_{2i}$$

- Difference score compounds the errors

$$\begin{aligned} d_i &= y_{2i} - y_{1i} \\ &= (\tau_{2i} + \varepsilon_{2i}) - (\tau_{1i} + \varepsilon_{1i}) \\ &= (\tau_{2i} - \tau_{1i}) + (\varepsilon_{2i} - \varepsilon_{1i}) \end{aligned}$$

compounded error typically exceeds error in either repeated measure



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Controversy in Measuring Change

- The potentially high unreliability of raw difference scores led to much hand wringing and some rather strong statements that these should never be used…

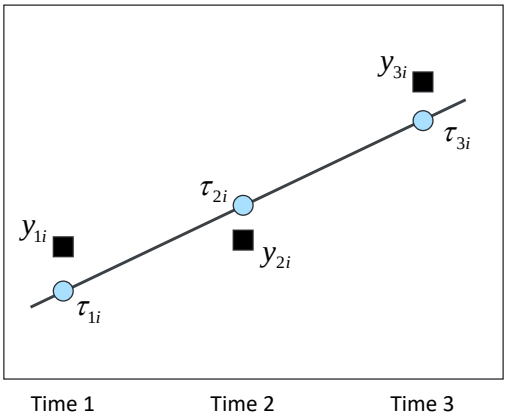
*"**Raw change**...scores formed by subtracting pretest scores from posttest scores lead to fallacious conclusions... the unsuitability of such scores has long been discussed, they are still employed, even by some otherwise sophisticated investigators."* (Cronbach & Furby, 1970)

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Cronbach, L., & Furby, L. (1970). How should we measure 'change'- Or should we? *Psychological Bulletin*, *74*, 68-80.

Inferring Latent Change

- However, with a 3rd time point, can fit a line through observed values to help dissociate underlying change from error
  - Assuming linear growth, but nonlinear possible with more repeated measures
  - Here, true scores follow line exactly but don't have to; we will allow for true-score variation around line which can be predicted via time-varying covariates
- This is the essence of a latent curve
  - Inferring a latent trajectory from observed repeated measures



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Summary

- Raw change is intuitive first step in capturing individual change
- Much controversy given inherent unreliability in change score
- Can overcome this problem given availability of 3+ time points from which to infer true underlying change
- Latent curve model leverages multiple repeated measures to infer underlying latent change over time
  - avoids use of potentially unreliable raw-score change

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For further reading on these topics, see:

Hendrix, L. J., Carter, M. W., & Hintze, J. L. (1979). A comparison of five statistical methods for analyzing pretest–posttest designs. *Journal of Experimental Education, 47*, 96–102

Huck, S. W., & McLean, R. A. (1975). Using a repeated measures ANOVA to analyze the data from a pretest–posttest design: A potentially confusing task. *Psychological Bulletin, 82*, 511–518.

Maxwell, S.E. & Delaney, H.D. (2004). *Designing experiments and analyzing data: A model comparison perspective*. New York: Taylor & Francis.

Rausch, J.R., Maxwell, S.E. & Kelley, K. (2003). Analytic methods for questions pertaining to a randomized pretest, posttest, follow-up design. *Journal of Clinical and Consulting Psychology, 32*, 467-486.

Rogosa, D. R. (1995). Myths and methods: "Myths about longitudinal research," plus supplemental questions. In J. M. Gottman (Ed.) *The Analysis of Change* (pp. 3-65). Hillsdale, New Jersey: Lawrence Erlbaum.

Willett, J.B. (1997). Measurement of Change. In J. P. Keeves (Ed.), *Educational Research, Methodology and Measurement: An International Handbook, 2nd. Edition*. Oxford, England: Pergamon Press.

Willett, J.B., Singer, J.D., and Martin, N.C. (1998). The design and analysis of longitudinal studies of development and psychopathology in context: Statistical models and methodological recommendations. *Development and Psychopathology, 10*, 395-426.

Chapter Summary

- Many advantages to collection and analysis of longitudinal data
- A key goal is to typically to examine between-person differences in within-person change
- Raw difference score is an intuitive option, but controversial given potential unreliability problems
- With 3+ time points, can differentiate underlying true change (at latent level) from raw change (at observed level)
  - this is the heart of the latent curve model
- LCMs exceptionally well suited to evaluating questions about inter-individual differences in intra-individual change

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Chapter 2
The Unconditional Linear Latent Curve Model

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2.3 Thinking More Closely About Time 2-25
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2.1 Defining a Latent Growth Curve

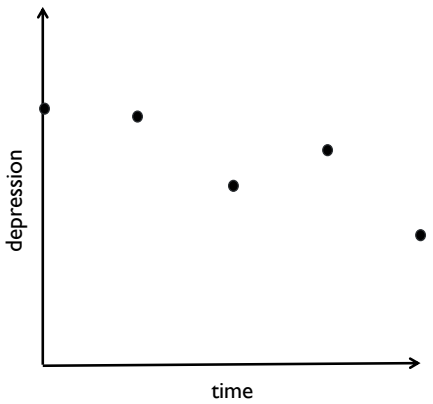
Objectives

- Introduce general concept of a latent growth curve
- Describe a growth curve for a single individual
- Describe a set of growth curves for multiple individuals
- Differentiate mean trajectory from individual variability around the mean trajectory

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Repeated Measures for One Person

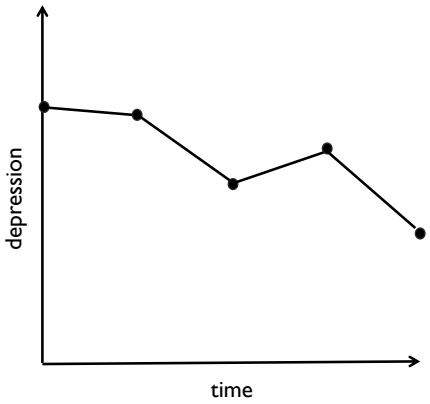
- Consider hypothetical case where we have five repeated measures assessing depression in a single adolescent



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Repeated Measures for One Person

- Could connect observations to see time-adjacent changes
  - would represent a series of difference scores

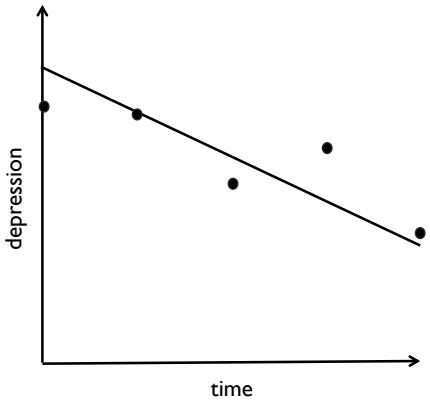


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Given a 1-unit difference on time, the slope for any given segment would equal the difference score between the two observations.

A Growth Curve for One Person

- Could instead "smooth over" repeated measures and estimate a continuous trajectory for this individual via line of best fit

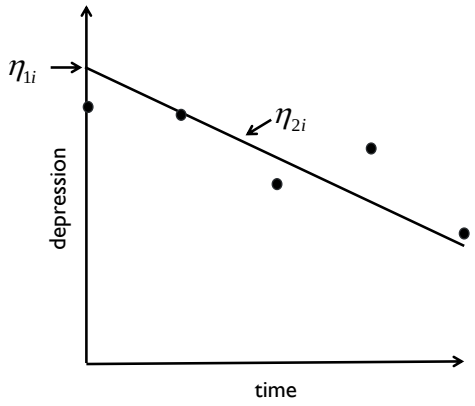


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If we were to fit a line to just two assessments, not only would it fit perfectly (because we are simply connecting the two dots) but the slope of the line would capture *raw change* between the two points (i.e., the 2nd assessment minus the 1st). With three or more assessments, we *over identify* the line and will get a test of fit to the observed data.

A Growth Curve for One Person

- Can summarize straight-line trajectory by two pieces of information
  - an intercept (η_{1i}) and slope (η_{2i}) unique to individual i

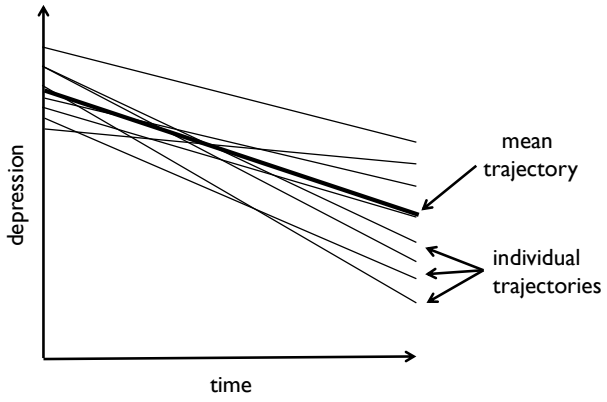


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We use the η_{1i} and η_{2i} to denote intercept and slope so that later latent curve models correspond with the standard notation used in the general SEM.

Growth Curves for Multiple Persons

- Rarely interested in just one individual – usually have a sample of many
- can extend to 8 trajectories
  - but would use 100+ in practice
- can also consider the mean and variance of the collection of individual trajectories



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The Latent Growth Curve

- Trajectory means
  - the average value of the parameters that define the growth trajectory pooling over all individuals in the sample
- Trajectory variances
  - the variability of individual cases around the mean trajectory parameters
  - larger variances reflect greater variability in growth
- Can consider various restrictions on these parameters to model different patterns of growth over time

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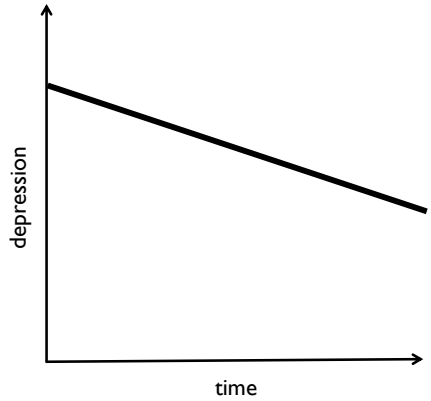
In the multilevel modeling framework, the means are referred to as *fixed effects* and the variability around the means as *random effects*. There are many close ties between the multilevel and SEM approaches to growth modeling. For example, see:

Willett, J. B., & Sayer, A. G. (1994). Using covariance structure analysis to detect correlates and predictors of change. *Psychological Bulletin, 116*, 363-381.

See also our YouTube playlist that compares multilevel modeling to latent curve modeling approaches: https://www.youtube.com/playlist?list=PLQGe6zcSJT0VxMZUN6DBuhIoCRZNoA2Vz

No Variance for Intercept or Slope

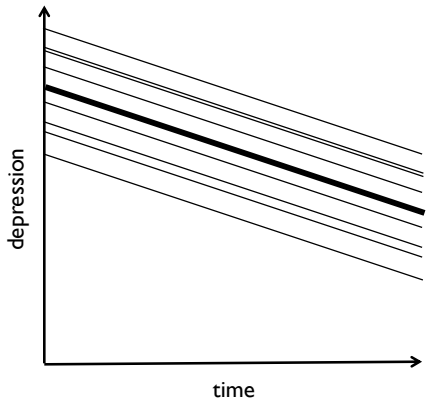
- Implies all individual trajectories are stacked precisely on top of one another in a single line



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Mean Slope, Mean & Variance Intercept

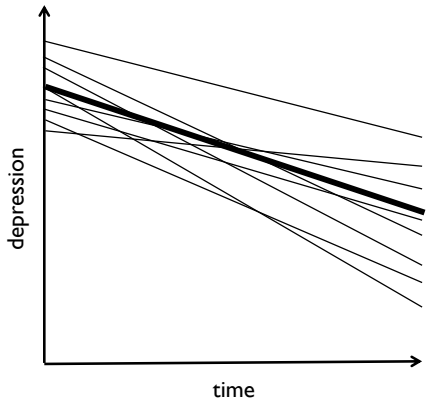
- Implies individual variability in starting point but constant rate of change over time



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Mean & Variance both Intercept & Slope

- Implies individual variability in starting point and rate of change



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Initial Motivating Questions

- What is the mean course of change over time?
  - linear, quadratic, exponential, no change, etc.
- Are there individual differences in the course of change?
  - variability in starting point and rate of change
- Are there time-invariant or time-varying predictors of change?
  - person-level characteristics like biological sex, ethnicity diagnostic status
  - time-specific characteristics like onset of drinking, arrest, marriage
- Do two constructs travel together through time?
  - trajectories of alcohol use linked with trajectories of depression

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We will see how we can evaluate each of these questions as we proceed through the class. In this chapter, we focus on the first two with a focus on linear models. We then extend to nonlinear models in Chapter 3. Chapter 4 considers time-invariant predictors, whereas Chapter 5 considers time-varying predictors and multivariate growth models. Chapter 6 then shows a number of potential extensions on these models.

Summary

- Initial motivation is that there exists a true underlying continuous trajectory of change that was not directly observed
- Want to use the observed repeated measures to infer existence of underlying latent trajectory
- Means capture overall values of parameters that define growth trajectory
- Variances capture individual variability in parameters that define growth trajectory
- Goal is to build a model that incorporates predictors of individual variability in growth
- Will build this model using the SEM analytical framework

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2.2 Latent Growth Curves as a Confirmatory Factor Model

Objectives

- Briefly review confirmatory factor analysis (CFA) model
- Define latent curve as a special type of CFA
- Explore mean and covariance structures implied by the LCM

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Capturing Growth as a Latent Factor

- Theory posits existence of unobserved continuous trajectory
- Cannot directly observe, but can infer existence based on set of repeated measures
- Growth curve fits naturally into the SEM-based latent variable model
  - e.g., depression, self esteem, worker productivity, etc.
- Can draw on strengths of confirmatory factor analysis (CFA) to define latent curve model
- The LCM is fundamentally a highly restricted CFA model

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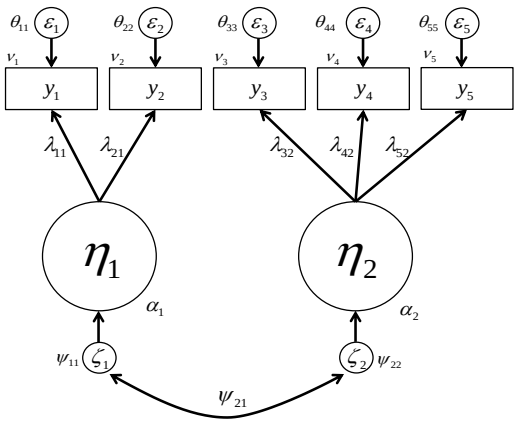
Confirmatory Factor Analysis

- Primarily theory-driven: test model that specifies the number and nature of the latent factors behind set of observed measures
  - e.g., latent depression and anxiety underlie set of 20 symptom items
- Model identified through restrictions on parameters
- Number of latent factors determined by theory
- Factor pattern matrix is restricted by analyst to reflect theory
  - e.g., some loadings freely estimated, others fixed to zero
- Attention paid to global and local fit of model to data

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Here we briefly review the CFA, assuming prior exposure to this model. For those who would like additional coverage of CFA, this is a core component of our free workshop on structural equation modeling, available at https://centerstat.org/structural-equation-modeling/

The Parameters of the General CFA



y	observed item
ε	item residual
θ	residual variance
ν	item intercept
λ	factor loading
η	factor score
α	factor mean
ζ	factor disturbance
ψ	factor co/variance

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There is no single method for creating path diagrams. Here we use the convention that rectangles are observed variables and circles are unobserved (or latent) variables. We do not use shapes to denote means or intercepts, but instead use symbols to denote the parameter adjacent to the corresponding variable (i.e., ν) or factor (i.e., α). We will not always carry this level of diagrammatic detail forward.

The Measurement Equation

- Measurement equation for p-observed variables and k-latent factors is

$$\mathbf{y}_i = \boldsymbol{\nu} + \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i = \begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{pi} \end{bmatrix} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_p \end{bmatrix} + \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{k1} \\ \lambda_{21} & \cdots & \lambda_{k2} \\ \vdots & \cdots & \vdots \\ \lambda_{p1} & \cdots & \lambda_{kp} \end{bmatrix} \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \\ \vdots \\ \eta_{ki} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \vdots \\ \varepsilon_{pi} \end{bmatrix}$$

- The residuals are distributed as $\boldsymbol{\varepsilon}_i \sim MVN(\mathbf{0}, \boldsymbol{\Theta})$ where

$$\boldsymbol{\Theta} = \begin{bmatrix} \theta_{11} & & & \\ 0 & \theta_{22} & & \\ 0 & 0 & \ddots & \\ 0 & 0 & 0 & \theta_{pp} \end{bmatrix}$$

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Normality assumptions on the errors are not always necessary, depending on the method of estimation. Maximum likelihood estimation, which we use here, explicitly invokes this assumption, but will nevertheless provide accurate estimates under many common conditions and model fit and parameter estimates can be corrected for non-normality to provide robust inferences. See Chapter 6 for details.

The Structural Equation

- Structural equation for k-latent factors is defined as

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i = \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \\ \vdots \\ \eta_{ki} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} + \begin{bmatrix} \zeta_{1i} \\ \zeta_{2i} \\ \vdots \\ \zeta_{ki} \end{bmatrix}$$

- The deviations are distributed $\boldsymbol{\zeta}_i \sim MVN(\mathbf{0}, \boldsymbol{\Psi})$ where

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_{11} & & & \\ \psi_{21} & \psi_{22} & & \\ \vdots & & \ddots & \\ \psi_{k1} & \psi_{k2} & \cdots & \psi_{kk} \end{bmatrix}$$

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See above note regarding assumption of normality, which applies equally with respect to the disturbances of the latent factors.

The Reduced-Form Equation

- Can substitute structural equation into measurement equation to obtain the *reduced-form* equation

$$\mathbf{y}_i = \boldsymbol{\nu} + \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$$

$$\mathbf{y}_i = \boldsymbol{\nu} + \boldsymbol{\Lambda}\left(\boldsymbol{\alpha} + \boldsymbol{\zeta}_i\right) + \boldsymbol{\varepsilon}_i$$

- Because we now express observed measures in $\mathbf{y}_i$ solely as a function of model parameters, can define mean and covariance structure among observed measures as implied by the model

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In the standard CFA model, the mean structure is typically saturated. In other words, there are usually as many means estimated as were observed in the sample, and thus the mean structure does not contribute to the overall fit of the model. For this reason, many presentations of the CFA model omit discussion of the mean structure entirely. However, the mean structure becomes critically important when transitioning from a standard CFA to the LCM. As we will see in a moment, the mean structure for the LCM is not typically saturated and thus fewer means are estimated than were observed.

Model-Implied Moment Structures

- Motivating goal of CFA is to define model that reproduces mean and covariance structure of observed data as closely as possible
- Begin by defining a vector theta that contains all of the parameters that define the CFA, say

$$\boldsymbol{\theta}' = \begin{pmatrix} \nu_1 & \nu_2 & \cdots & \lambda_{11} & \lambda_{22} & \cdots & \psi_{11} \end{pmatrix}$$

- Use this vector to compute a *matrix-valued function* that expresses the means and covariances among our observed measures in $\mathbf{y}_i$ solely as a function of model parameters
  - model-implied mean structure denoted $\boldsymbol{\mu}(\boldsymbol{\theta})$
  - model-implied covariance structure denoted $\boldsymbol{\Sigma}(\boldsymbol{\theta})$

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Model-Implied Moment Structures

- The reduced-form expression for the CFA is

$$\mathbf{y}_i = \boldsymbol{\nu} + \boldsymbol{\Lambda}\left(\boldsymbol{\alpha} + \boldsymbol{\zeta}_i\right) + \boldsymbol{\varepsilon}_i$$

- The associated model-implied mean and covariance structure is

$$\boldsymbol{\mu}\left(\boldsymbol{\theta}\right) = E\left(\mathbf{y}_i\right) = E\left[\boldsymbol{\nu} + \boldsymbol{\Lambda}\left(\boldsymbol{\alpha} + \boldsymbol{\zeta}_i\right) + \boldsymbol{\varepsilon}_i\right] = \boldsymbol{\nu} + \boldsymbol{\Lambda}\boldsymbol{\alpha}$$

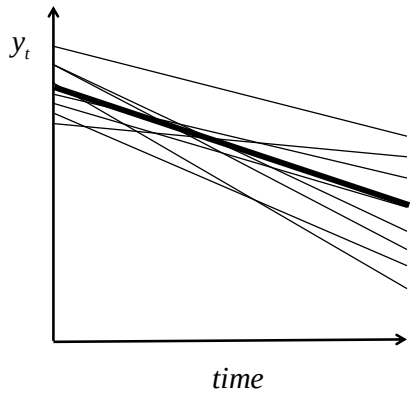
$$\boldsymbol{\Sigma}\left(\boldsymbol{\theta}\right) = \text{var}\left(\mathbf{y}_i\right) = \text{var}\left[\boldsymbol{\nu} + \boldsymbol{\Lambda}\left(\boldsymbol{\alpha} + \boldsymbol{\zeta}_i\right) + \boldsymbol{\varepsilon}_i\right] = \boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}' + \boldsymbol{\Theta}$$

- Goal of estimation to select values of $\boldsymbol{\theta}$ that make mean and covariance structure implied by model as close as possible to the mean and covariance structure observed in the sample

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Returning to the Latent Curve

- We believe individual trajectories exist for each case, but these were not observed directly
  - want to infer the latent curves based on the data that were observed



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Equation for Individual Curves

- Begin by expressing repeated measures for a given individual as additive function of underlying linear trajectory weighted by *time*

$$y_{ti} = \eta_{1i} + \eta_{2i} time_t + \varepsilon_{ti}$$

- For example, consider three time points:

$$y_{1i} = \eta_{1i} + \eta_{2i} time_1 + \varepsilon_{1i}$$
$$y_{2i} = \eta_{1i} + \eta_{2i} time_2 + \varepsilon_{2i}$$
$$y_{3i} = \eta_{1i} + \eta_{2i} time_3 + \varepsilon_{3i}$$

- Note that the intercept (η_{1i}) is implicitly weighted by the value of 1, and the slope (η_{2i}) is weighted by the numerical value of *time*

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Equation for Individual Curves

- If we place each term in the scalar equations into matrices:

$$\mathbf{y}_i = \begin{bmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \end{bmatrix} \quad \mathbf{\Lambda} = \begin{bmatrix} 1 & time_1 \\ 1 & time_2 \\ 1 & time_3 \end{bmatrix} \quad \boldsymbol{\eta}_i = \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} \quad \boldsymbol{\varepsilon}_i = \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \varepsilon_{3i} \end{bmatrix}$$

then we can equivalently express our set of scalar equations as:

$$\mathbf{y}_i = \mathbf{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \end{bmatrix} = \begin{bmatrix} 1 & time_1 \\ 1 & time_2 \\ 1 & time_3 \end{bmatrix} \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \varepsilon_{3i} \end{bmatrix} = \begin{bmatrix} \eta_{1i} + \eta_{2i} time_1 + \varepsilon_{1i} \\ \eta_{1i} + \eta_{2i} time_2 + \varepsilon_{2i} \\ \eta_{1i} + \eta_{2i} time_3 + \varepsilon_{3i} \end{bmatrix}$$

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Whereas in the multilevel growth model the numerical value of time is recorded within the data set as a variable that is used as an explicit predictor of the repeated measures, in the latent curve model the numerical value of time is not represented as an exogenous variable and is instead entered via specification of the factor loadings for the slope factor.

The Measurement Equation

- Note similarity of LCM equation:

$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

to CFA equation from earlier:

$$\mathbf{y}_i = \boldsymbol{\nu} + \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

- Indeed, measurement equation for the LCM is the same as the measurement equation for the CFA with the restriction that

$$\boldsymbol{\nu} \equiv \mathbf{0}$$

  - item-specific intercepts are restricted to zero and LCM must reproduce sample means entirely through means of latent factor

- We next need to define a model for the latent curve factors

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The Structural Equation

- Recall we can write the intercept and slope as a function of the mean and individual deviation from the mean:

$$\eta_{1i} = \alpha_1 + \zeta_{1i}$$
$$\eta_{2i} = \alpha_2 + \zeta_{2i}$$

- We can again compactly summarize these in matrix form

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$$

$$\begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \zeta_{1i} \\ \zeta_{2i} \end{bmatrix} = \begin{bmatrix} \alpha_1 + \zeta_{1i} \\ \alpha_2 + \zeta_{2i} \end{bmatrix}$$

- This expression precisely matches that of the CFA

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The LCM Equations

- General expression for the measurement equation for a linear trajectory with $t=1,2,...,T$ repeated measures is $\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{Ti} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & T-1 \end{bmatrix} \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \vdots \\ \varepsilon_{Ti} \end{bmatrix}$$

  - numerical value of *time* enters model via factor loading matrix
- The structural equation is $\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$
  - vectors are same length as number of latent factors that define trajectory

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Here we define the loading for the slope factor to be equal to one less than the associated value of time. This results in the first assessment being set equal to zero and in turn defines the intercept as the mean of the trajectory at the first occasion. We explore the coding of time in much greater detail in a moment.

LCM Model-Implied Moment Structures

- The model-implied mean and covariance structure for the LCM:

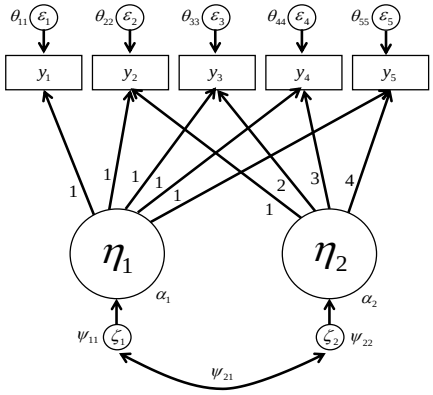
$$\boldsymbol{\mu}(\boldsymbol{\theta}) = \boldsymbol{\Lambda}\boldsymbol{\alpha}$$

$$\boldsymbol{\Sigma}(\boldsymbol{\theta}) = \boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}' + \boldsymbol{\Theta}$$

- Precisely same expressions as CFA except no item intercepts
- Also precisely same goal of estimation to select values of $\boldsymbol{\theta}$ that make mean and covariance structure implied by model as close as possible to mean and covariance structure observed in sample
- We can graphically represent LCM as a path diagram

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A Linear LCM as a Path Diagram



$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$$

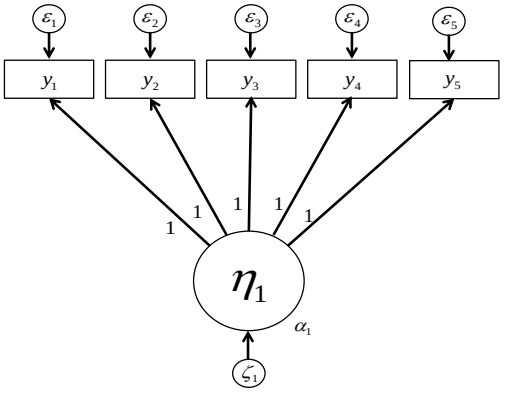
- We can examine different expressions of this model...

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In the standard CFA model described earlier, typically some or all of the factor loadings are freely estimated. In contrast, in the LCM all of the factor loadings are usually fixed to pre-defined values. This is a unique characteristic of the latent curve model relative to the CFA. Although the factor loadings are entirely fixed in many LCMs, we will see in Chapter 3 that we can potentially still estimate some factor loadings as one way to model nonlinear trajectories of change over time.

Intercept-only LCM

- Implies outcome does not change as function of time
  - all individual trajectories horizontal but at different levels



$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \\ y_{4i} \\ y_{5i} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [\eta_{1i}] + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \varepsilon_{3i} \\ \varepsilon_{4i} \\ \varepsilon_{5i} \end{bmatrix}$$

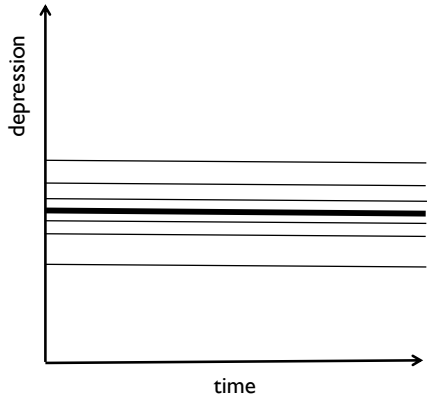
$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$$

$$[\eta_{1i}] = [\alpha_1 + \zeta_{1i}]$$

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Intercept-only LCM: Trajectories

- Intercept-only LCM implies between-person variability in overall level of outcome, but outcome does not change with time



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Intercept-only LCM: Factor Mean

- Just one factor mean because only one factor is defined to represent latent intercept

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$$

$$E(\boldsymbol{\eta}_i) = \boldsymbol{\alpha}$$

$$\boldsymbol{\alpha} = [\alpha_1]$$

- This simply reflects the mean level of all repeated measures pooled over all individuals

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Intercept-only LCM: Factor Variance

- Just one factor variance again because only one factor is defined to represent latent intercept

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$$

$$\boldsymbol{\zeta}_i \sim N(\mathbf{0}, \boldsymbol{\Psi})$$

$$\boldsymbol{\Psi} = [\psi_{11}]$$

- This represents the individual variability around the overall mean

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Intercept-only LCM: Residual Variance

- Finally, time-specific residuals for RMs allowed to obtain a unique value at each time point and are typically uncorrelated over time

$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

$$\boldsymbol{\varepsilon}_i \sim N(\mathbf{0}, \boldsymbol{\Theta})$$

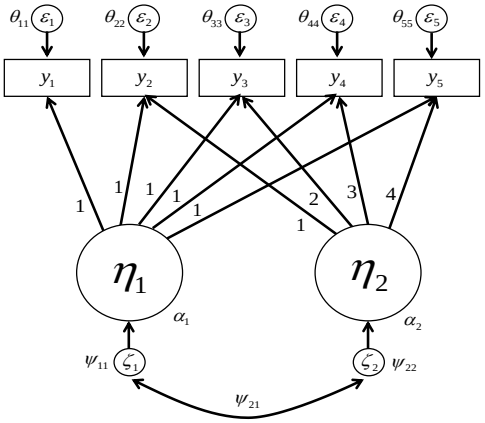
$$\boldsymbol{\Theta} = \begin{bmatrix} \theta_{11} & & & & \\ 0 & \theta_{22} & & & \\ 0 & 0 & \theta_{33} & & \\ 0 & 0 & 0 & \theta_{44} & \\ 0 & 0 & 0 & 0 & \theta_{55} \end{bmatrix}$$

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Here we estimate a unique residual variance at each time period; this is called *heteroscedasticity*. In a moment we will compare this to a model in which we estimate a single residual variance for all time periods; this is called *homoscedasticity*.

Linear LCM

- Can add a second correlated factor to capture linear change:



$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \\ y_{4i} \\ y_{5i} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \varepsilon_{3i} \\ \varepsilon_{4i} \\ \varepsilon_{5i} \end{bmatrix}$$

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$$

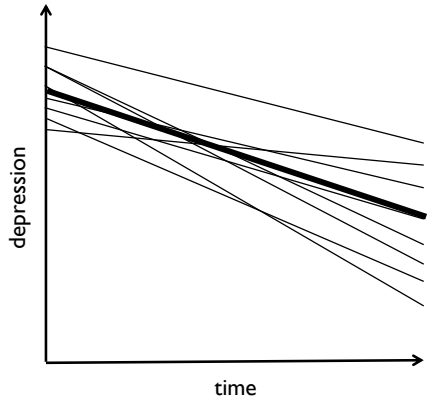
$$\begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} = \begin{bmatrix} \alpha_1 + \zeta_{1i} \\ \alpha_2 + \zeta_{2i} \end{bmatrix}$$

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In this chapter we focus explicitly on the linear growth model. We will expand this to a variety of nonlinear functions in Chapter 3.

Linear LCM: Trajectories

- Intercept and linear slope model implies individual differences in both level and rate of change



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Linear LCM: Latent Factor Mean Structure

- Two factor means, one for intercept and one for linear slope

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$$

$$E(\boldsymbol{\eta}_i) = \boldsymbol{\alpha}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

- Reflects mean starting point and mean linear rate of change
  - defines starting point because we set first value of time to be zero
  - we will explore this in much greater detail shortly

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Linear LCM: Latent Factor Variance Structure

- Factor variance now expressed as a covariance matrix with variance of intercept and slope and covariance between the two

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$$

$$\boldsymbol{\zeta}_i \sim N(\mathbf{0}, \boldsymbol{\Psi})$$

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_{11} & \\ \psi_{21} & \psi_{22} \end{bmatrix}$$

- Represents individual variability around starting point and rate of change, and covariance between starting point and rate of change

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Linear LCM: Item Variance Structure

- Interestingly, the covariance structure among the time-specific residuals is precisely the same as with the intercept-only LCM

$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

$$\boldsymbol{\varepsilon}_i \sim N(\mathbf{0}, \boldsymbol{\Theta})$$

$$\boldsymbol{\Theta} = \begin{bmatrix} \theta_{11} & & & & \\ 0 & \theta_{22} & & & \\ 0 & 0 & \theta_{33} & & \\ 0 & 0 & 0 & \theta_{44} & \\ 0 & 0 & 0 & 0 & \theta_{55} \end{bmatrix}$$

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Heteroscedastic Residuals

- Standard LCM assumes each time-specific repeated measure is defined by a unique residual variance
  - called heteroscedasticity
- A simplifying condition is to assume error variances are equal
  - called homoscedasticity

$$\boldsymbol{\Theta} = \begin{bmatrix} \theta_{11} & & & & \\ 0 & \theta_{22} & & & \\ 0 & 0 & \theta_{33} & & \\ 0 & 0 & 0 & \theta_{44} & \\ 0 & 0 & 0 & 0 & \theta_{55} \end{bmatrix}$$

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Homoscedastic Residuals

- Can impose equality constraint on residuals over time
- This is a testable hypothesis
  - homoscedasticity more parsimonious, but may not correspond to characteristics of the observed data
- Typically want to identify most parsimonious structure that does not significantly contribute to model misfit

$$\boldsymbol{\Theta} = \begin{bmatrix} \theta & & & & \\ 0 & \theta & & & \\ 0 & 0 & \theta & & \\ 0 & 0 & 0 & \theta & \\ 0 & 0 & 0 & 0 & \theta \end{bmatrix}$$

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Communality: Time-specific R^2

- Can use time-specific residuals to estimate proportion of observed variance in repeated measure that is accounted for by the latent factors
  - in traditional factor analysis, called *communality* or h^2
  - usually called r-squared in SEM
- Simply complement of ratio of residual variance to total variance

$$R_t^2 = 1 - \frac{\text{var}(\varepsilon_{ti})}{\text{var}(y_{ti})} = 1 - \frac{\theta_{tt}}{\sigma_{y_t}^2}$$

- Values are interpreted in the usual way
  - higher values reflect stronger associations between items and factors
  - good standardized effect sizes to report in practice

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Summary

- The latent curve model fits logically within the CFA
- Estimate a latent factor for each trajectory component
- Enter numerical measure of time via factor loading matrix
- Can build basic to complex models and test intervening steps
- Can compare homo- vs. heteroscedasticity
- Can allow for correlated residuals if needed

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2.3 Thinking More Closely About Time

Objectives

- More closely consider role of time in the LCM with regard to
  - choosing the ideal metric of time
  - where to place the zero point
  - the unequal spacing of time
  - individually-varying assessments of time

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The Numerical Value of Time

- The numerical value of time enters LCM via the factor loading matrix:

$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{Ti} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & T-1 \end{bmatrix} \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \vdots \\ \varepsilon_{Ti} \end{bmatrix}$$

- Traditionally, numerical value of time reflected *wave of assessment*
  - but in many applications, other metrics of time may be more relevant
- A key alternative metric of time to consider is *chronological age*

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When Age and Wave are Equivalent

Age at Assessment Period

wave 1	*wave* 2	*wave* 3	*wave* 4
6	7	8	9

- When wave of assessment & age are equivalent, it does not matter which metric of time is used in the growth model
- Say want to scale intercept to represent outcome at start of trajectory
  - *wave* = *wave* - 1; *age* = *age* - 6; both result in times of 0, 1, 2, 3

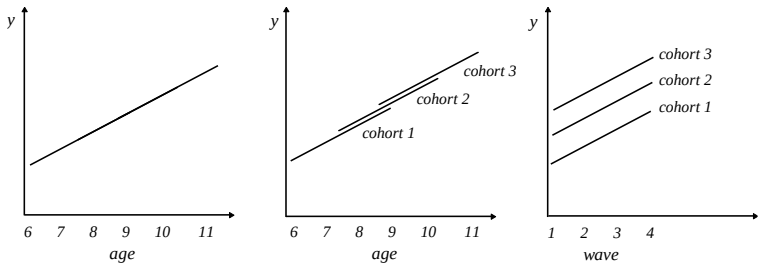
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This design would be characteristic of a birth-cohort study in which all children born at a given time are assessed at wave 1 and followed over time. As such, age and wave are isomorphic.

When Age and Wave are Different

Age at Assessment Period

cohort	*wave* 1	*wave* 2	*wave* 3	*wave* 4
1	6	7	8	9
2	7	8	9	10
3	8	9	10	11



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This is a more common situation encountered in passive observational designs in which there is age heterogeneity within each wave of assessment. This may be accidental or by design (e.g., accelerated longitudinal designs). If age is the important metric of time yet wave is used in the analysis, significant bias can result.

Reorganizing Data on Alternative Time Metric

- Four assessments results in six unique ages, and many advantages in working with more repeated observations
- But reorganization of time creates a missing data problem
  - that is, although in example all subjects provided data at all four waves, missing data are introduced *by design*
- We must consider methods of estimation that allow for partially missing data

	chronological age					
cohort	6	7	8	9	10	11
1	×	×	×	×	•	•
2	•	×	×	×	×	•
3	•	•	×	×	×	×

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We can easily reorganize the data by age instead of wave of assessment. Now it is a missing data problem, although we must invoke additional assumptions.

Sidebar on Missing Data

- Missingness can arise in longitudinal data for many reasons:
  - Different time points for different people such that no one observed at all time points (e.g., cohorts in accelerated longitudinal design)
  - Intermittent missingness / non-response
  - Attrition (nearly universal problem in longitudinal studies)
- Can be of three types:
  - Missing Completely At Random (MCAR): any missing value is strictly random and unrelated to any other variables in or out of the model
  - Missing At Random (MAR): any missing value can be related to variables in the model, but not to variables omitted from the model
  - Missing Not At Random (MNAR): any missing value is related to variables that are omitted from the model

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For different possible time metrics, the MAR (or even MCAR) assumption may be quite reasonable, particularly if this variation was introduced by design as in the multiple cohort structure above. Attrition and non-response are the missing data processes where you start to worry about MNAR, which is challenging to address. Fortunately, unless MNAR missingness is prevalent and strongly informative in the data, continuing to assume MAR likely won't lead you too far astray.

The LCM and Missing Data

- ML estimation of the SEM (and thus the LCM) allows for partially missing data under assumption of MAR
  - More flexible treatment of missing data than classical methods like repeated measures ANOVA
- Missingness often at least approximately MAR
  - Even if drop-out is related to outcome, missing values may still be predictable from those that *were* observed (e.g., a person with high and increasing depression drops out but we already know where they were heading given their early repeated measures)
- More complex approaches for MNAR go beyond the scope of this class
  - selection models, pattern mixture models, etc.

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Note that missing data on any exogenous predictors in an LCM will result in deletion of observations under the usual conditional likelihood that is used to fit the model. We will see in our software demonstrations, however, that we can "trick" SEM programs into using a joint likelihood instead, so that exogenous variables are treated like endogenous variables, allowing for partial missingness even on the predictor side.

Examples of Alternative Metrics of Time

Outcome Variable	**Operant Time Variable**
Academic Outcomes	Grade of School
Employee Productivity	Years Employed in Position
Cognitive Decline in Elderly	Years to Death
Response to Drug	Days of Treatment
Deviant Peer Affiliation	Pubertal Status

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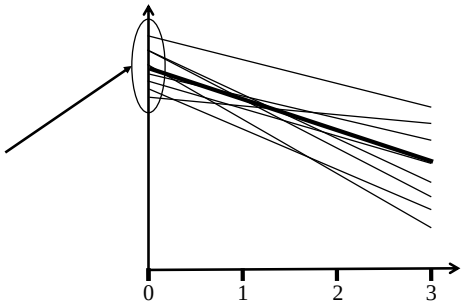
Many options for measuring time can be considered as a function of the outcome under study.

Coding Time: Zero Point

- Up to this point we have set numerical value of time to equal zero at the first assessment period:

$$\lambda_t = 0, 1, 2, 3$$

Mean and variance of intercept is scaled for initial time period because when time is zero the slope factor does not contribute



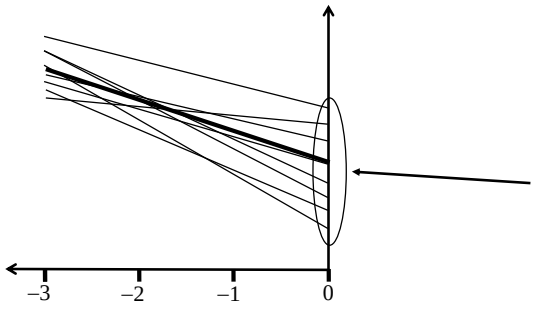
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Note that λ_t denotes the numerical scaling of *time* and appears in the second column of the factor loading matrix for a linear LCM (where the first column consists of all 1's to define the intercept factor).

Coding Time: Zero Point

- Can code time such that the intercept defines the *last* assessment

$$\lambda_t = -3, -2, -1, 0$$



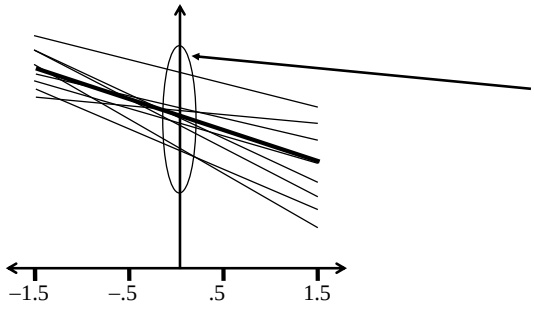
Mean and variance of intercept scaled to reflect the last time period; might be useful if conducting a treatment evaluation

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Coding Time: Zero Point

- Can code time so the intercept defines the *middle* assessment

$$\lambda_t = -1.5, -.5, .5, 1.5$$



Mean and variance of intercept scaled to reflect the middle time period

With even number of assessments, middle may not be directly observed but is implied by model

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Coding Time: Zero Point

- Alternative coding schemes for time are implemented via factor loadings

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & -3 \\ 1 & -2 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1.5 \\ 1 & -0.5 \\ 1 & 0.5 \\ 1 & 1.5 \end{bmatrix}$$

- All of these result in same model fit, but several properties of LCM change depending on the numerical coding of time

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Coding Time: Zero Point

- What does change
  - the mean of the intercept factor
  - the variance of the intercept factor
  - the covariance between the intercept and slope factor
  - effects of predictors of the intercept factor
- What does not change
  - Model fit
  - the mean of the linear slope factor
  - the variance of the linear slope factor
  - effects of predictors of the slope factor
- *Disclaimer*: The above holds for a linear trajectory and things get more complicated for nonlinear models yet to come

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The above summary holds for the linear LCM. Some complications arise in coding of time when considering higher-order polynomial functions (e.g., quadratic or cubic). For details, see:

Biesanz, J.C., Deeb-Sossa, N., Aubrecht, A.M., Bollen, K.A., & Curran, P.J. (2004). The role of coding time in estimating and interpreting growth curve models. *Psychological Methods, 9*, 30-52.

Coding Time: Spacing

- Thus far have assumed equally spaced assessments
  - e.g., one year increment between each assessment
- No reason to be limited to equally spaced assessments
  - can have some or even all assessments unequally spaced
- For example, say observe ages 6, 7, 10, 12, & 16
  - factor loadings scaled relative to first two assessments
  - matrix to right shows subtracting 6 from all ages
- Placement of the zero point holds here as well
  - can again put zero at beginning, middle, or end

$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 4 \\ 1 & 6 \\ 1 & 10 \end{bmatrix}$$

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Coding Time: Unit-of-Time

- We also control the unit of time under study
  - e.g., assessments taken at 6, 12, 18, 30, and 42 months
- Can scale time in months, half-years, or full-years

$$\Lambda_{month} = \begin{bmatrix} 1 & 0 \\ 1 & 6 \\ 1 & 12 \\ 1 & 24 \\ 1 & 36 \end{bmatrix} \quad \Lambda_{half-year} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} \quad \Lambda_{full-year} = \begin{bmatrix} 1 & 0 \\ 1 & .5 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

- All models fit identically, but mean and variance of slope factor scaled in the given unit of time

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Both the choice of zero-point, spacing, and unit of time can be controlled with a single linear transformation equation; see Bollen & Curran (2006), pages 115-120.

Bollen, K.A., & Curran, P.J. (2006). *Latent Curve Models: A Structural Equation Approach*. John Wiley & Sons: New Jersey.

Coding Time: Unit-of-Time

- What potentially <u>does</u> change
  - mean and variance of slope factor
  - covariance of intercept with slope factor
  - point estimates of any parameter estimates associated with slope
    - note comment to right: the *p*-values of predictors do <u>not</u> change
- What definitely does <u>not</u> change
  - model fit
  - mean and variance of intercept factor
  - effects of predictors of intercept factor
  - *p*-values of any parameter estimates associated with slope
    - *numerical values rescaled to different units-of-time, but <u>same</u> p-value*
- *Same disclaimer*: The above holds for a linear trajectory and things get more complicated for nonlinear models yet to come

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Individually-Varying Measures of Time

- LCM allows for different assessment schedules across individuals
  - e.g., some assessed at ages 11, 12, & 14 and others at ages 10, 13, & 16, etc.
- Standard LCM requires each age-specific outcome be a manifest variable
  - e.g., observed measures of outcome at ages 11, 12, 13, 14, 15 & 16
- Fundamentally a missing data problem
  - subjects assessed at age 10 and 12 are "missing" at age 11
  - subjects assessed at age 12 and 14 are "missing" at age 13
  - but combined sample has observations at all ages
- As long as missingness not MNAR, can be handled in LCM through application of ML estimator

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Individually-Varying Measures of Time

- Some longitudinal designs have highly variable assessments
  - e.g., daily diary data where subjects randomly pinged throughout day
- At extreme, could have no two people provide assessments at same time
- Standard LCM not well suited for data such as these
  - must be able to compute mean and variance of RM at each time period
- Must instead use *definition variable methodology* in LCM
  - defines a unique factor loading matrix for each person based on person-specific time scores, then fits LCM aggregating over individual matrices
- Places the LCM much closer to a multilevel modeling approach
  - indeed, MLM may be a better analytic strategy

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Boker, S., Neale, M., Maes, H., Wilde, M., Spiegel, M., Brick, T., ... & Fox, J. (2011). OpenMx: an open source extended structural equation modeling framework. *Psychometrika, 76*, 306-317.

Neale, M. C., Aggen, S. H., Maes, H. H., Kubarych, T. S., & Schmitt, J. E. (2006). Methodological issues in the assessment of substance use phenotypes. *Addictive Behaviors, 31*, 1010-1034.

Summary

- Can select metric of time that is ideal for question at hand and select numerical values of time to achieve different goals
  - Can define alternative zero-points for time (e.g., beginning, middle, or end)
  - Can define unequal assessment intervals (e.g., 6 month, 12 month, 24 month)
  - Can alter metric of change (e.g., days, weeks, months, etc.)
  - Can incorporate individual measures of time via definition variables
- All centered around ultimate goal of identifying optimal functional form of change over time

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2.4 Demonstration: Trajectories of Antisocial Behavior

Objectives

- Introduce real data set studying trajectories of child and adolescent antisocial behavior from ages 6 to 13
- Fit series of models to repeated measures of antisocial behavior to identify optimally fitting LCM
- Draw initial conclusions about developmental course of antisocial behavior prior to including predictors of level and change

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Example: Antisocial Behavior

- N=405 cases drawn from National Longitudinal Survey of Youth
- Age 6-8 at first assessment; assessed up to 3 more times every-other year
- Mother report of child antisocial behavior on six items, each with a 0,1,2 response scale. Sum score ranges from 0 to 12
- Initial research questions:
  - what is the optimal mean functional form of the trajectory over time?
  - is there individual child-to-child variability around these mean values?
- Later we will consider child-specific predictors of the trajectories

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Example: Antisocial Behavior

- Accelerated longitudinal cohort design where *age* is time metric
- Accelerated cohorts result in missingness introduced by design
  - "x" denotes data that are present ; "•" denotes data that are missing

cohort	6	7	8	9	10	11	12	13
1	x	•	x	•	x	•	x	•
2	•	x	•	x	•	x	•	x
3	•	•	x	•	x	•	x	•
N	122	168	146	192	151	174	135	173

- Design allows for an *eight year* age span even though no individual child was assessed more than *four times*
  - Total N=405 children, 221 of whom have complete data at all 4 time points

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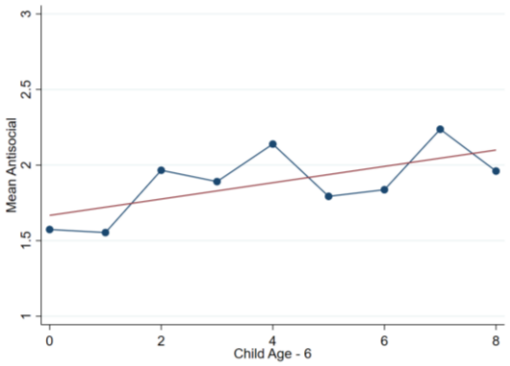
Motivating Questions

- Is there evidence of individual variability in the overall level antisocial behavior?
- Does the inclusion of a linear slope factor significantly improve model fit?
- Is there evidence of individual variability in starting point and rate of change over time?
- Does the model support homoscedasticity for the time-specific residuals?
- Does the final model adequately fit the observed data?

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Antisocial Behavior Age-Specific Means

- Note that although any individual case has between one and four repeated assessments, there are eight unique ages
  - we can consider these summary statistics both in a plot and a table



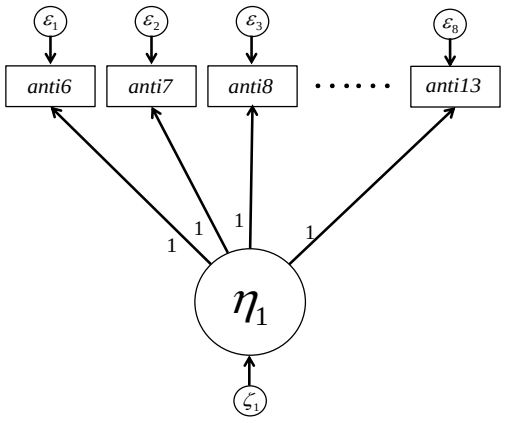
Age	*N*	*Mean*	*Variance*
6	122	1.57	2.76
7	168	1.60	2.41
8	146	1.90	3.34
9	192	1.99	3.83
10	151	2.01	4.48
11	174	1.91	3.95
12	135	1.88	3.54
13	173	2.38	5.25

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These are maximum likelihood estimates that account for missing data. Simple statistics based only on the available data at each age would be slightly different than these values.

Intercept-Only Model

- We will begin by estimating an intercept-only model
  - although we expect this to fit poorly given the increasing sample means

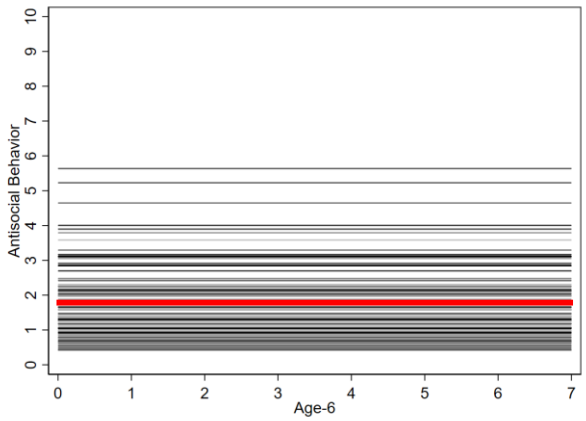


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We do not expect this model to fit well given the observed increases in the means over time. But this is a good model with which to begin and to use as a comparison to later model expansions.

Model-Implied Trajectories

- Latent mean and variance both significantly different from zero
  - mean=1.87, variance=1.68, both *p*'s<.0001
- Model implies horizontal trajectories distributed around latent mean
  - 100 estimated individual trajectories shown, with mean trajectory indicated in red
- However, we must of course consider model fit



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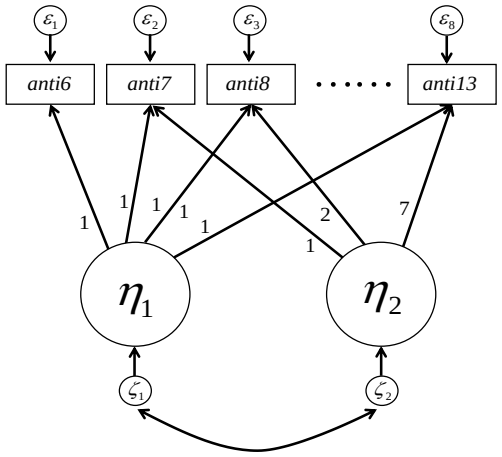
Intercept-Only Model

- As expected, this model fit the data quite poorly
  - $\chi^2(31) = 95.55$, $p < .0001$
  - CFI=.79
  - TLI=.83
  - RMSEA=.07
  - SRMR=.16
- No index supports adequate fit of the model to the sample data
- Raises the question: would the addition of a slope factor improve fit?
  - because the intercept-only is nested within the linear, we can conduct an LRT to formally compare the fit of the two models to the sample data

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Intercept and Linear Slope Model

- We added a slope factor to the intercept-only model



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Intercept and Slope Model

- Fit indices reflects that linear model fits data rather well
  - $\chi^2(28) = 42.88$, $p = .04$;
  - CFI=.95; TLI=.96;
  - RMSEA=.036;
  - SRMR=.12
- Model fit data significantly better compared to intercept-only:
  - intercept-only: $\chi^2(31) = 95.55$
  - intercept + slope model: $\chi^2(28) = 42.88$
  - LRT difference: $\chi^2(3) = 52.67$ ($p < .0001$)
- Will retain slope given improvement in model fit

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As is often found in latent curve models, the SRMR is elevated relative to commonly used cut-offs denoting good fit (e.g., .08). It is not entirely clear why LCMs tend to have elevated SRMR values.

Intercept and Slope Model

- Model defined to have heteroscedastic residuals
  - all age-specific residuals take on unique values
- Model could be simplified if single value estimated for all ages
  - prefer models with fewer parameters as they are *parsimonious*
- This is *homoscedasticity* and is a testable hypothesis
  - will conduct an LRT to compare heteroscedastic vs. homoscedastic

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Testing for Homoscedasticity

- Prior model defined *heteroscedastic* errors: diagonal elements differ

$$\boldsymbol{\Theta} = \begin{bmatrix} 1.74 & & & & & & & \\ 0 & 1.27 & & & & & & \\ 0 & 0 & 1.64 & & & & & \\ 0 & 0 & 0 & 1.85 & & & & \\ 0 & 0 & 0 & 0 & 2.16 & & & \\ 0 & 0 & 0 & 0 & 0 & 1.71 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.11 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.62 \end{bmatrix}$$

- Can impose *equality constraint* on residuals and conduct an LRT to evaluate potential decrement in model fit.

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Given these residual variance estimates are bouncing around non-systematically, we might speculate that the differences in their values reflect only sampling error, and thus would not result in a significant improvement in fit above what one might expect by chance alone. The LRT formally evaluates this possibility.

Comparing Two Error Structures

- Heteroscedastic

$$\boldsymbol{\Theta} = \begin{bmatrix} 1.74 & & & & & & & \\ 0 & 1.27 & & & & & & \\ 0 & 0 & 1.64 & & & & & \\ 0 & 0 & 0 & 1.85 & & & & \\ 0 & 0 & 0 & 0 & 2.16 & & & \\ 0 & 0 & 0 & 0 & 0 & 1.71 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.11 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.62 \end{bmatrix}$$

- Homoscedastic

$$\boldsymbol{\Theta} = \begin{bmatrix} 1.71 & & & & & & & \\ 0 & 1.71 & & & & & & \\ 0 & 0 & 1.71 & & & & & \\ 0 & 0 & 0 & 1.71 & & & & \\ 0 & 0 & 0 & 0 & 1.71 & & & \\ 0 & 0 & 0 & 0 & 0 & 1.71 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.71 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.71 \end{bmatrix}$$

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Formally Testing Difference Between Models

- Prior model with heteroscedastic errors:

$$\chi^2\left(df = 28\right) = 42.88$$

- Model with equality constraints imposing homoscedasticity:

$$\chi^2\left(df = 35\right) = 50.07$$

- Resulting LRT is non-significant.

$$\chi^2_{\Delta}\left(df = 7\right) = 7.19,\ p = .66$$

- Supports retaining the homoscedasticity as most parsimonious model.

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Model Results Including Homoscedasticity

- The final model fit the data quite well

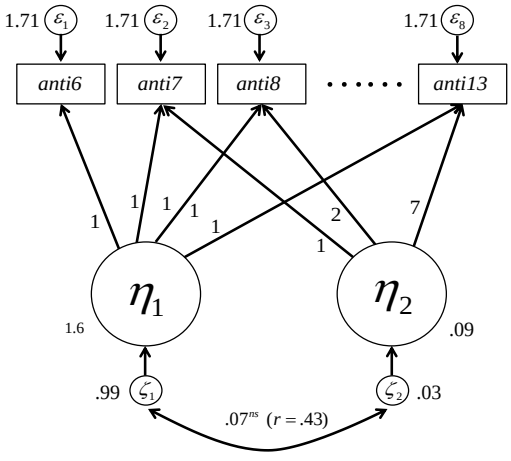
$$\chi^2\left(df = 35\right) = 50.07,\ p = .048$$
$$CFI = .95$$
$$TLI = .97$$
$$RSMEA = .033$$
$$SRMR = .12$$

- Often easiest to see final parameter estimates as part of a path diagram

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Final LCM Estimates

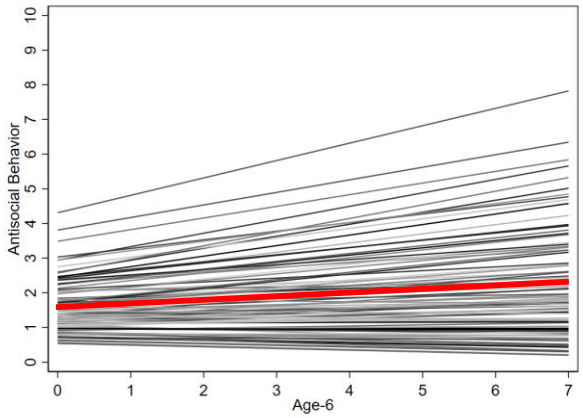
- All parameters are significant except for factor covariance



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Estimated Individual Trajectories

- We can again obtain estimates of first 100 individual trajectories
- Can clearly see much variability in starting point and rate of change
  - mean trajectory shown in red
- These plots are exceedingly useful in research reports to augment tables of numerical results



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These individual trajectories are sometimes called "empirical Bayes estimates / predictors" or "factor score estimates", and a collection of trajectories is often referred to as a "spaghetti plot" (for obvious reasons). These types of plots are invaluable when trying to communicate complex model results to a broad audience of readers.

Summary

- Fitting a series of nested latent curve models showed linear model with homoscedastic residuals fit well
- The mean trajectory indicated that, on average, adolescents reported significant increases in antisocial behavior over time
- There was significant individual variability in both the intercept and slope indicating that some subjects reported higher vs. lower initial values and increased at slower vs. faster rates over time
- Final unconditional model suggests that one or more predictors of growth could provide insight into what type of child starts higher and increases more slowly over time

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2.5 Frequently Asked Questions

Objectives

- Questions to be addressed here:
  - What if my estimated LCM fits poorly?
  - Can you specify more complex covariance structures for the residuals?
  - Can you estimate growth trajectories using individual regression models separately for each subject?
  - How does the LCM compare to the multilevel growth model?

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What if My Estimated LCM Fits Poorly?

- Because the LCM is a type of SEM, we have all the usual indices of fit
  - chi-square test statistic, fit indices, modification indices, etc.
- A poorly fitting LCM is interpreted in precisely the same was as any SEM
  - the model structure does not adequately reproduce the sample variances, covariances, and/or means
- Poor fit implies that we have defined a misspecified model
  - the sample structure does not correspond to the population structure
  - undermines our confidence in interpreting estimates or other results
- The model may either need to be modified or discarded, the decision of which depends on the theory, the sample, and the measures

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Wu, W., West, S. G., & Taylor, A. B. (2009). Evaluating model fit for growth curve models: Integration of fit indices from SEM and MLM frameworks. *Psychological Methods*, *14*, 183.

For a general discussion of the evaluation of fit in SEMs, see our blog post: https://centerstat.org/how-do-i-know-if-my-structural-equation-model-fits-the-data-well/

Can You Include More Complex Error Structures?

- For many applications, particularly with panel data, reasonable to assume latent factors fully account for correlations among repeated measures
  - like factor analysis, time-specific residuals assumed to be independent
- However, there are some situations in which we might expect the residuals around a person's trajectory to correlate over time
  - For example, observations sampled closely together in time such that a "shock" or perturbation at one time point reverberates through subsequent time points
- In such cases, can consider various serial correlation structures
  - e.g., auto-regressive or ARMA
- These more complex residual error structures can be included in the LCM (depending on options available in a given software package)

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We have to be careful in specifying residual structures that allow for residuals to correlate within-person over time, such an autoregressive residual structure. Doing so in effect sets up a competition between the within-person residual structure and the between-person differences in the latent curve factors to explain the same observed correlations in the repeated measures. If there is no theoretical reason to expect correlated residuals, including these might mask the omission of an important component of the latent trajectories (e.g., a quadratic component). Thus, we generally only recommend including some kind of residual correlation structure in the event that this is theoretically expected, such as with repeated measures gathered in close proximity.

For more discussion of residual structures see

Grimm, K. J., & Widaman, K. F. (2010). Residual structures in latent growth curve modeling. *Structural Equation Modeling, 17*, 424-442.

Kwok, O. , West, S. G. and Green, S. B. (2007) . The impact of misspecifying the within-subject covariance structure in multiwave longitudinal multilevel models: A Monte Carlo study. *Multivariate Behavioral Research, 42*, 557 – 592 .

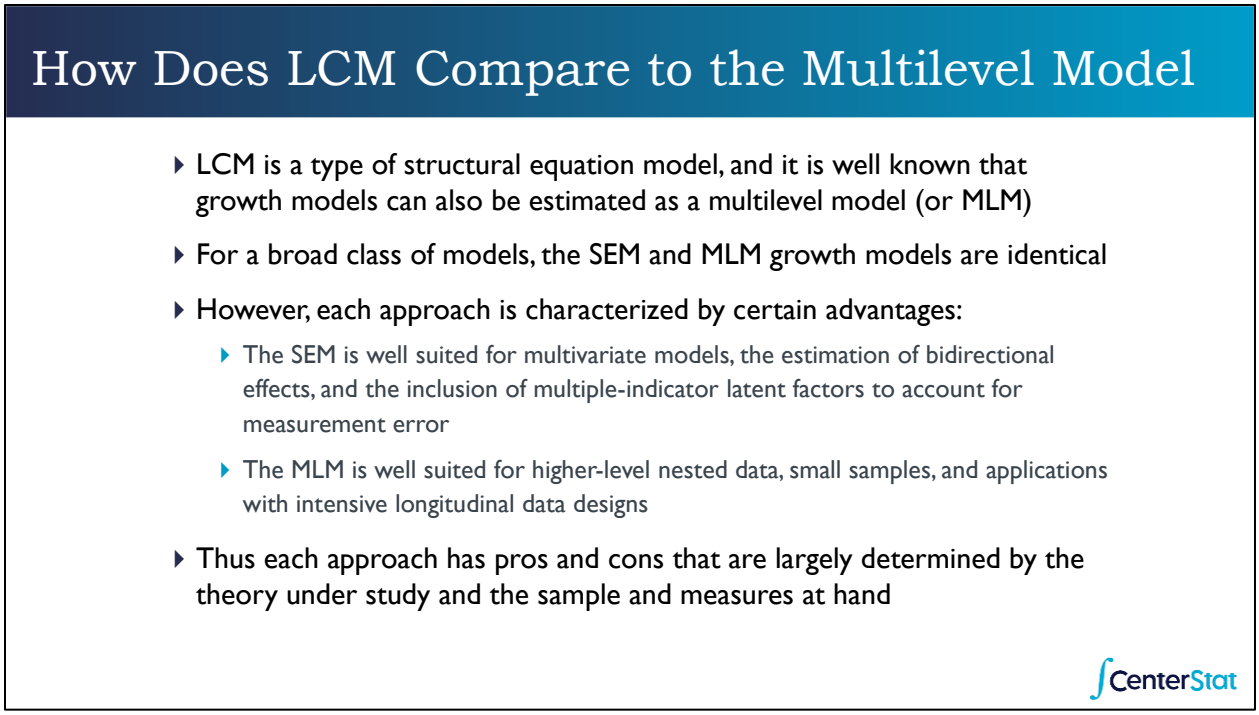
Can Trajectories Be Estimated Using Regression?

- Historically, it was proposed that individual growth trajectories could be estimated for each subject separately using the multiple regression model
  - once estimated, the individual intercepts and slopes were gathered together in a single data file for subsequent analysis
  - although quite logical, this approach is no longer recommended
- The reason is that although point estimates of the intercept and slope can be obtained using regression, rarely is *variability* considered
  - e.g., two slopes could both equal .5, but for one the repeated measures are tightly clustered around the line and for the other they are widely spread out
- The omission of variability can in turn bias all aspects of later analyses, and this is not recommended for use in practice
  - see Bollen & Curran (2005) for details

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Carrig, M. M., Wirth, R. J., & Curran, P. J. (2004). A SAS macro for estimating and visualizing individual growth curves. *Structural Equation Modeling, 11*, 132-149.

Rogosa, D., & Saner, H. (1995). Longitudinal data analysis examples with random coefficient models. *Journal of Educational and Behavioral Statistics, 20*, 149-170.



Chou, C. P., Bentler, P. M., & Pentz, M. A. (1998). Comparisons of two statistical approaches to study growth curves: The multilevel model and the latent curve analysis. Structural Equation Modeling: A Multidisciplinary Journal, 5(3), 247-266.

Willett, J. B., & Sayer, A. G. (1994). Using covariance structure analysis to detect correlates and predictors of individual change over time. *Psychological Bulletin*, *116*, 363.

See also our YouTube playlist that compares multilevel modeling to latent curve modeling approaches: https://www.youtube.com/playlist?list=PLQGe6zcSJT0VxMZUN6DBuhIoCRZNoA2Vz

Summary

- It is critical that adequate fit be established for any LCM so that the resulting parameter estimates can be interpreted with confidence
- Although the LCM often supports conditional independence of time-specific residuals, more complex structures are available if needed
- Individual trajectories can be estimated using traditional multiple regression, but this is not recommended in practice
- Growth models can be fit both within SEM framework (LCM) and within multilevel modeling framework
  - many core models can be fit equivalently in either framework
  - but each framework allows for unique extensions relative to the other

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Chapter Summary

- Goal of LCM is to estimate inter-individual differences in intra-individual change over time
- Believe latent trajectories gave rise to repeated measures, but trajectories not directly observed
- Use SEM framework to estimate underlying latent trajectories
- Define latent factor for each component of trajectory
- Can control zero-point, spacing, and metric of time
- Ultimately want to identify optimal functional form of growth over time so that can expand to more complex models

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Chapter 3
Nonlinear Latent Curve Models

3.1 Modeling Nonlinear Trajectories 3-3

3.2 Demonstration: Seasonal Trajectories of Crime 3-16

3.3 Frequently Asked Questions 3-25

3.1 Modeling Nonlinear Trajectories

Objectives

- Review implications of linear trajectory
- Describe three methods for estimating nonlinear trajectories
  - quadratic function
  - piecewise linear
  - freed factor loading
- Compare and contrast three approaches

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Implications of Linear Trajectories

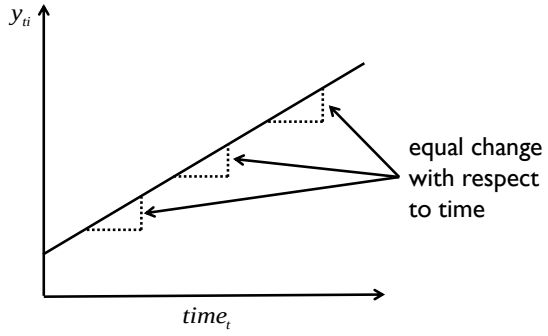
- Thus far we have only considered linear trajectories of change
- Linearity implies equal change in the outcome per unit-change of time regardless of where in time that occurs
- If slope is estimated to be .50, then a one-unit change in age is associated with a .50-unit change in the outcome whether considering age 11 to 12, or 21 to 22, or 31 to 32
- Change is thus **constant** with respect to time

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Polynomial of Degree 1: Linear

- A linear trajectory is a *polynomial of degree 1*:

$$y_{ti} = \eta_{1i} + \lambda_t \eta_{2i} + \varepsilon_{ti}$$



- Model-implied change is *constant* across values of time (λ_t)

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Measurement & Structural Equations

- For linear LCM, the measurement and structural equations are:

$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{Ti} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & T-1 \end{bmatrix} \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \vdots \\ \varepsilon_{Ti} \end{bmatrix}$$

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$$

$$\begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \zeta_{1i} \\ \zeta_{2i} \end{bmatrix}$$

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Nonlinear Trajectories

- Linear change can serve as reasonable approximation to modest nonlinear trend over time
- But change may not be constant with respect to time and linear trajectory would be a misspecified model
- We must consider alternative functions of time
- The first option is a higher-order polynomial

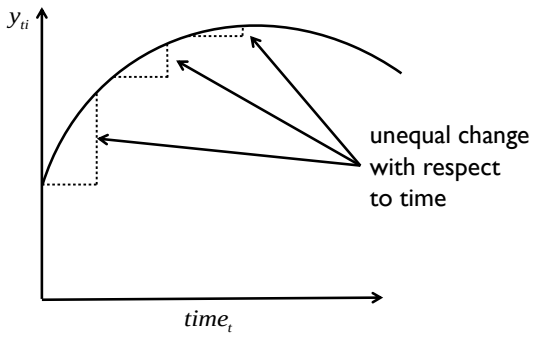
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Polynomial of Degree 2: Quadratic

- A quadratic trajectory is a *polynomial of degree 2*:

$$y_{ti} = \eta_{1i} + \lambda_t \eta_{2i} + \lambda_t^2 \eta_{3i} + \varepsilon_{ti}$$

- Model-implied change *varies* across values of time (λ_t)



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Quadratic trajectories tend work best when the observed means follow an approximately symmetric function over time.

Measurement & Structural Equations

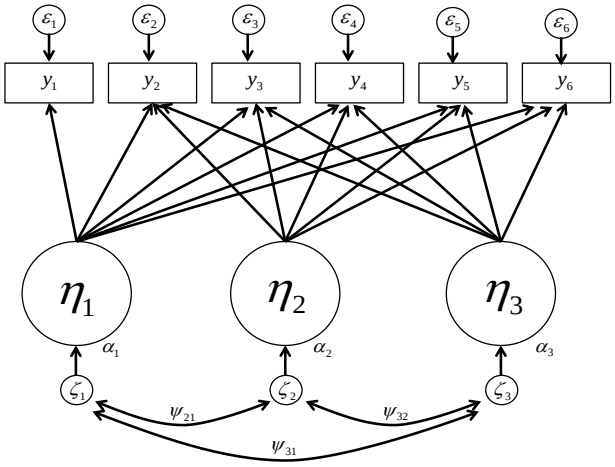
- For quadratic LCM, measurement and structural equations are

$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i \qquad \boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$$

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{Ti} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ \vdots & \vdots & \\ 1 & T-1 & (T-1)^2 \end{bmatrix} \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \\ \eta_{3i} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \vdots \\ \varepsilon_{Ti} \end{bmatrix} \qquad \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \\ \eta_{3i} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} \zeta_{1i} \\ \zeta_{2i} \\ \zeta_{3i} \end{bmatrix}$$

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Quadratic LCM



$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix}$$

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Interpreting Components of Quadratic

- Intercept: value of outcome where time equals zero
- Linear: slope of tangent line of curve where time equals zero
  - sometimes called the *first derivative* or *instantaneous velocity* at time=0
- Quadratic: change in slope of tangent line per unit change in time
  - sometimes called the *second derivative* or *acceleration*
- For linear trajectory, change is constant with respect to time
- For quadratic trajectory, change varies with respect to time
  - i.e., captures rate of change in change
- Allows for differential change at different points in time

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Technically, the quadratic parameter is equal to ½ the rate of acceleration, but it can still be loosely interpreted as an acceleration (or deceleration) parameter.

Quadratic and the Coding of Time

- Same issues hold for numerical coding of time as was discussed for linear LCM, but additional complexities must be considered

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & -3 & 9 \\ 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1.5 & 2.25 \\ 1 & -0.5 & .25 \\ 1 & 0.5 & .25 \\ 1 & 1.5 & 2.25 \end{bmatrix}$$

- All three models fit precisely the same
  - but with many time points, squared factor loadings can get excessively large and introduce estimation problems

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For further details about the coding of time in quadratic models, see:

Biesanz, J.C., Deeb-Sossa, N., Aubrecht, A.M., Bollen, K.A., & Curran, P.J. (2004). The role of coding time in estimating and interpreting growth curve models. *Psychological Methods, 9*, 30-52.

Piecewise Linear LCM

- Many advantages of quadratic model, but also several limitations
  - very restrictive and may fit observed data poorly
  - common to have small or even zero variance in quadratic factor
- Can approximate nonlinear function by tying together two or more linear pieces
  - called a *piecewise linear LCM* because jointly using two or more linear trajectories to approximate a more complex nonlinear one
- Can tie linear pieces together at a natural transition (e.g., children moving from middle to high school) or at a point on the curve that corresponds to an inflection point

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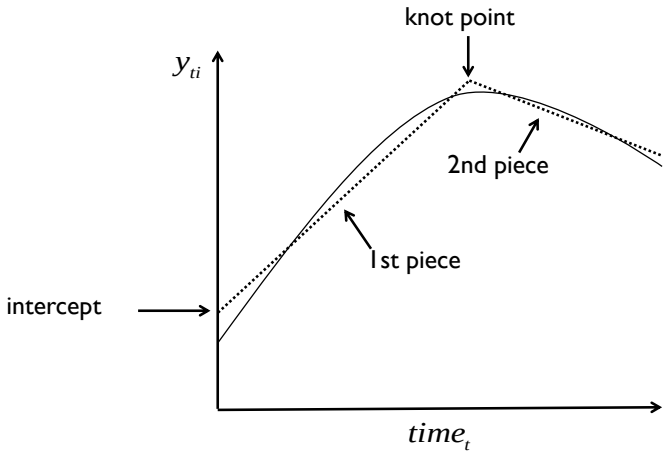
Piecewise Linear LCM

- Each linear piece defined by its own latent slope factor
- Some cases the knot point is logical
  - e.g., age at transition to middle school, shared by all individuals at same age
- Other cases is more of a subjective decision
  - examine pattern of means and identify inflection point of curve
- Sensitivity analyses can be conducted to compare different knots
  - e.g., fit models with knot at ages 11, 12 and 13 and compare results
- Once decide on optimal knot, must then select appropriate coding to define separate pieces

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Flora, D. B. (2008). Specifying piecewise latent trajectory models for longitudinal data. *Structural Equation Modeling, 15*, 513-533.

Piecewise Linear LCM



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Piecewise Linear LCM

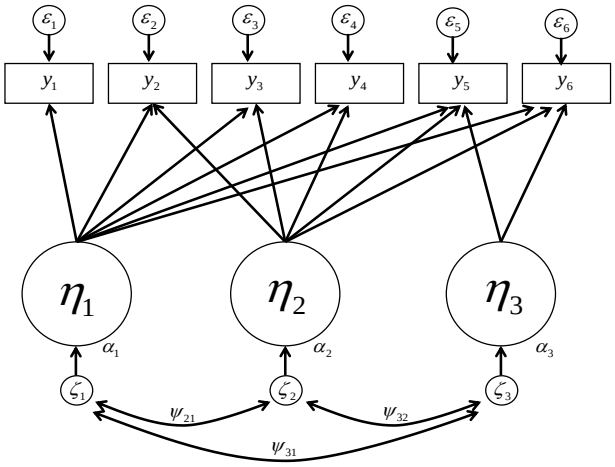
- Most common strategy is to define first piece as linear slope in the usual way, but to stop incrementing in value at the knot point
- The second piece then picks up incrementing after the knot point
- For example, for T=6 with a knot at the 4th time point:

$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

1st piece
knot point
2nd piece

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Piecewise Linear LCM



$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

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Piecewise Linear LCM

- General rule is each additional piece requires a latent factor
  - in principle can have as many pieces as data will support
- Three repeated measures needed to over-identify each linear piece
  - can share points with multiple pieces (e.g., $T = 5$ can define two linear pieces)
- Can combine linear and nonlinear pieces
  - e.g., first piece is linear and second piece is quadratic
- One piece can have estimated variance, one piece not

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Piecewise Linear LCM

- Standard linear LCM is nested within piecewise linear, so can obtain formal LRT to evaluate improvement in fit with 2nd piece
- Despite flexibility of approach, piecewise LCM still imposes a rather strict functional form of linearity within each piece
- One last option for nonlinearity is to not impose a functional form but estimate nonlinear change from the data
  - called a freed loading (or *latent basis*) LCM

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Freed Loading LCM

- The linear, quadratic, and piecewise models are all defined by fixing factor loadings to pre-defined values
- Key advantage is that subsequent interpretations are clear
  - can unambiguously interpret linear slope or quadratic curvature
- Key disadvantage is that repeated measures may not follow such a highly structured pattern of change over time
  - adequate reproduction of observed data might be difficult
- Can draw on strength of LCM: freely estimate certain loadings
- Lose ability to test specific parametric function, but gain substantial flexibility in modeling nonlinear change over time

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The freed loading model is also called the fully latent basis model. See, e.g.,

Meredith, W., & Tisak, J. (1990). Latent curve analysis. *Psychometrika, 55*, 107-122.

McArdle, J. J. (2009). Latent variable modeling of differences and changes with longitudinal data. *Annual Review of Psychology, 60*, 577-605.

Freed Loading LCM

- Say that we had 6 repeated observations that displayed some nonlinearity in the time-specific means
- We could estimate either a quadratic or a piecewise linear

$$\boldsymbol{\Lambda}_{quad} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix} \qquad \boldsymbol{\Lambda}_{pw} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

- Both require three latent factors, and both define all factor loadings to be equal to pre-defined values

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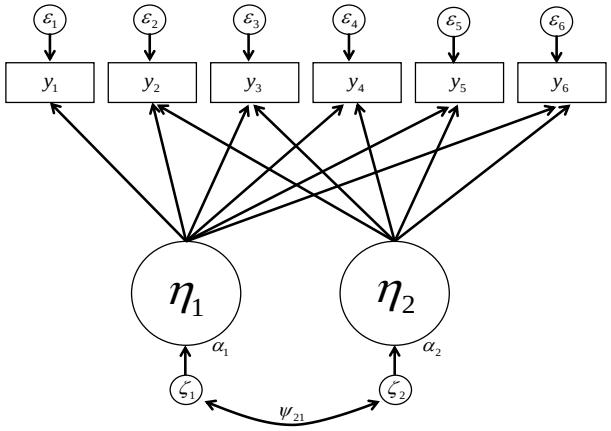
Freed Loading LCM

- We could instead define an intercept factor and add a "shape" factor on which some loadings are freely estimated and thus take on "optimal" values that correspond to observed data
  - Option 1 is to fix the first and second loadings to 0 and 1
  - Option 2 is to fix the first and last loadings to 0 and 1
- Both fit equivalently but have slightly different interpretations

$$\boldsymbol{\Lambda}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & \lambda_{32} \\ 1 & \lambda_{42} \\ 1 & \lambda_{52} \\ 1 & \lambda_{62} \end{bmatrix} \qquad \boldsymbol{\Lambda}_2 = \begin{bmatrix} 1 & 0 \\ 1 & \lambda_{22} \\ 1 & \lambda_{32} \\ 1 & \lambda_{42} \\ 1 & \lambda_{52} \\ 1 & 1 \end{bmatrix}$$

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Freed Loading LCM



$$\mathbf{\Lambda}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & \lambda_{32} \\ 1 & \lambda_{42} \\ 1 & \lambda_{52} \\ 1 & \lambda_{62} \end{bmatrix}$$

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Freed Loading Option #1

- By fixing the 1st and 2nd loadings to 0 and 1, all subsequent change is relative to that which occurred at 1st two time points
- Note linear model implies equal change at subsequent times: 0, 1, 2, 3, 4
- But loadings can be smaller or larger if nonlinear change exists
  - e.g., 0 and 1, then 1.9, 2.6, 3.3 -- decelerating rate of change
  - e.g., 0 and 1, then 2.2, 3.6, 5.7 -- accelerating rate of change
- Mean and variance of "shape" factor now represents propensity to change nonlinearly over time
- Linear LCM is nested within freed loading LCM, so LRT can test improvement in model fit with estimated factor loadings

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Freed Loading Option #2

- Fixing the first and last loadings to 0 and 1, all subsequent change is relative to the total change that occurred over all time points
- Could be accomplished with standard linear model: 0, .25, .50, .75, 1.0
- But loadings can be smaller or larger if nonlinear change exists
  - e.g., 0, .70, .85, .94, 1.0 -- decelerating rate of change
  - e.g., 0, .10, . 25, .50, 1.0 -- accelerating rate of change
- Mean and variance of "shape" factor again represents propensity to change nonlinearly over time
- Linear LCM again nested within freed loading LCM, so LRT can test improvement in model fit with estimated factor loadings

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To reiterate, Options #1 and #2 are *the same model*, just parameterized a bit differently. Either option generates the exact same fit. The parameter estimates are simply re-scaled to provide different interpretations.

Pros and Cons of Freed Loading LCM

- Many advantages
  - not restricted to highly structured predefined parametric functions
  - many social science constructs do not follow strict polynomials
  - draws on strengths of CFA to estimate optimal factor loadings
- But also potential disadvantages
  - "shape" factor no longer easily interpretable beyond "propensity to change nonlinearly over time"
  - can be overly flexible and potentially over-fit the data, especially means
  - can behave oddly when unstable or non-monotonic change
    - e.g., estimate loadings of 0, .5, .3, 1. 7, .8, 1.0

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There's nothing really wrong with loadings that go up then down – it just gets a bit trickier to interpret the loading estimates (e.g., the change between time 1 and time 4 was 170% of the change that between time 1 and time 6, reflecting a trajectory that went up then back down).

Summary

- Linear trajectory implies equal change per equal time interval regardless of where in time that occurs
- Many constructs do not change in this manner
- Nonlinear trajectory implies unequal change per equal time interval depending on where in that that occurs
- Three initial options for nonlinear LCM
  - quadratic, piecewise linear, and freed loading
- Other options available as well
  - e.g., reparameterized quadratic or structured latent curve

CenterStat

For an excellent overview of alternative approaches to modeling nonlinear functions, see

Cudeck, R., & Harring, J.R. (2007). Analysis of nonlinear patterns of change with random coefficient models. *Annual Review of Psychology, 58,* 615-637.

3.2 Demonstration: Seasonal Trajectories of Crime

Objectives

- Use data drawn from FBI files tracking criminal activity across a large number of reporting agencies in the North Eastern US
- Show that a linear LCM fits poorly
- Compare and contrast three nonlinear LCMs

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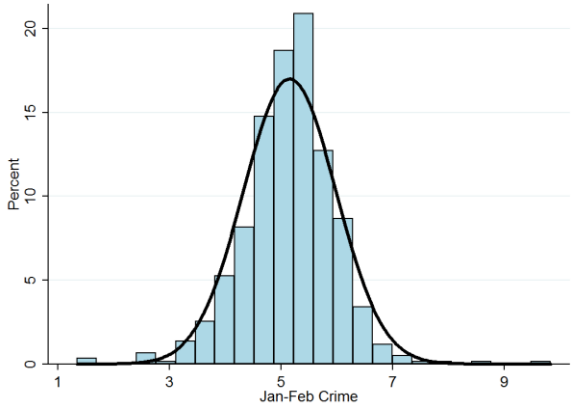
Uniform Crime Reports

- 1995 monthly Uniform Crime Reports (UCRs) obtained from National Archive of Criminal Justice
- Selected 626 reporting areas from the North Eastern US
- Computed mean of frequency of major crime occurrence
  - e.g., murder, robbery, assault, larceny, etc.
- Combined bi-months to increase stability of estimates
  - e.g., T=6 assessments of Jan/Feb, March/April, etc.
- Items continuous and scaled as crime per 100,000 population

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Uniform Crime Reports for Jan-Feb

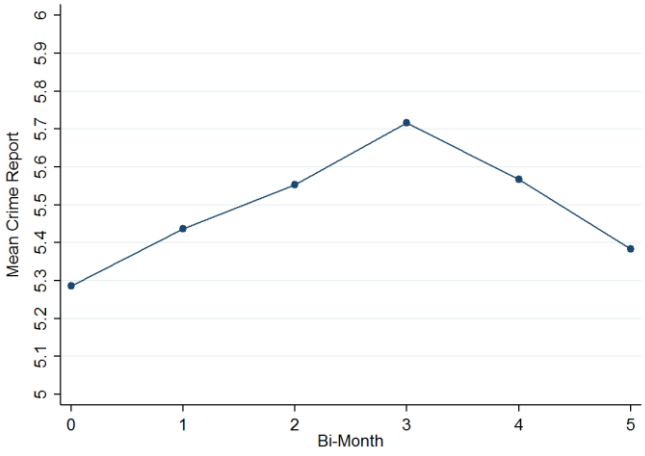
- We can examine the distributions of the UCR measures
  - we are assuming these are continuous and normal



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Plot of Time-Specific Means

- A plot of the time-specific means suggests clear nonlinearity



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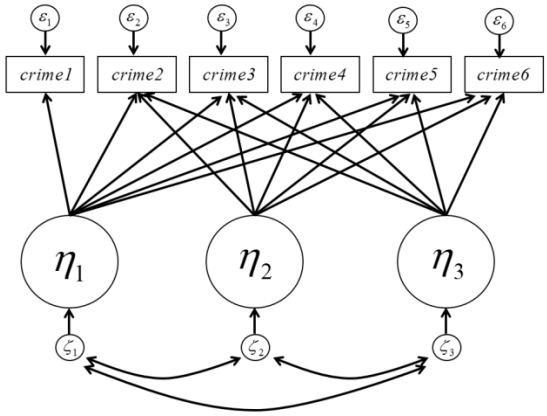
Model Building

- As before we began with a random intercept only model
  - as expected, this fit quite poorly: $\chi^2(df = 19) = 634.16,\ p < .0001$
- We expanded this to include a linear slope factor
  - this model also fit poorly: $\chi^2(df = 16) = 544.32,\ p < .0001$
- Even though the linear slope significantly improved model fit via the LRT, $\chi^2_\Delta(df = 3) = 89.84,\ p < .0001$, the model is unacceptable
- Indeed, the linear model resulted in the dreaded "`the latent variable covariance matrix (psi) is not positive definite`"
- We must consider an alternative model structure

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Quadratic Growth Model

- We expanded the linear model to include a quadratic factor
  - this added four parameters: latent mean and variance, and 2 covariances



$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix}$$

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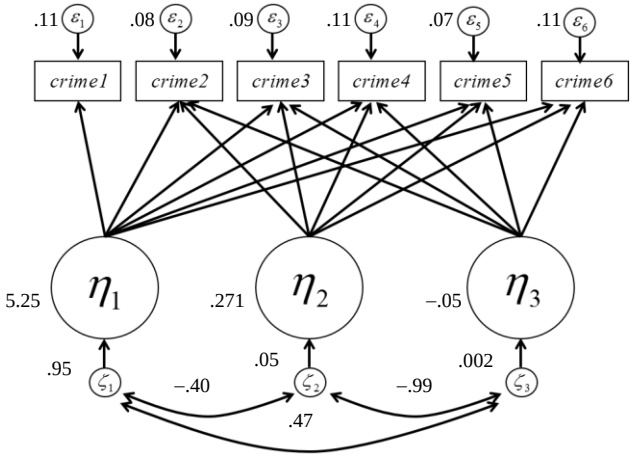
Quadratic Growth Model

- This model also fit the data quite poorly
  - $\chi^2(df = 12) = 132.15,\ p < .0001,\ RMSEA = .126$
- Although it did fit significantly better than the linear model
  - $\chi^2_\Delta(df = 4) = 412.17,\ p < .0001$
- Due to the extremely poor overall fit, we would normally not interpret the resulting parameters
- However, we show these here simply to highlight how the results might be interpreted in practice
- We present the empirical results using the path diagram

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Quadratic Growth Model

- All estimates are significant and covariances given as correlations



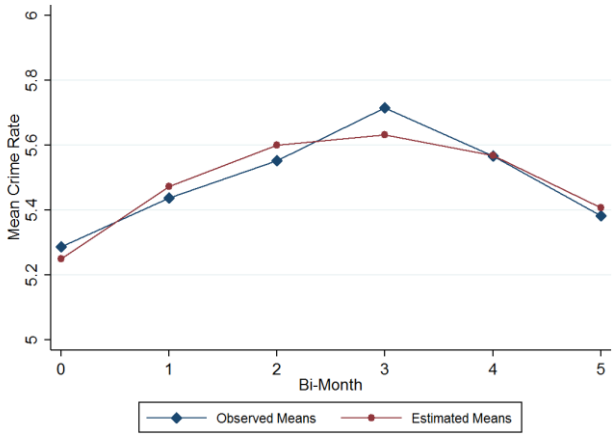
$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix}$$

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The model-implied starting point is 5.25 (the intercept); the linear rate of change at the initial time point is .27 (the velocity); and the change in the rate of change per unit-time is $-.1$ (the acceleration; the rate of change in the slope of the tangent line is $2 \times \eta_3$ and thus $2 \times -.05 = -.1$).

Quadratic Growth Model

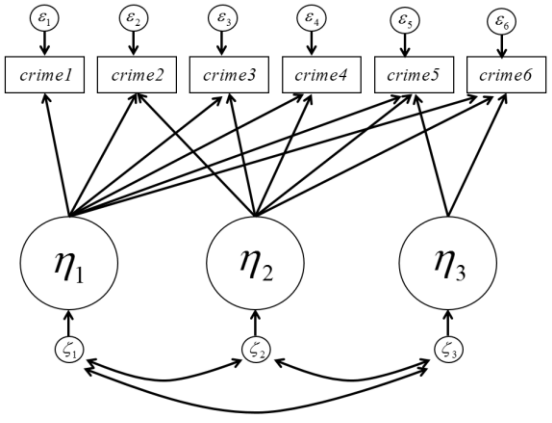
- Helpful to see model-implied trajectory super-imposed on the time-specific means
  - can see some means well replicated, others are not



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Piecewise Growth Model

- We next fit a piecewise growth model
  - same number of parameters as quadratic, but different loading matrix



$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

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Piecewise Growth Model

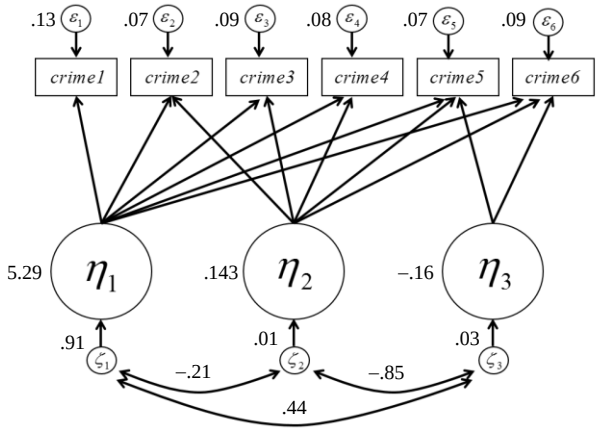
- The piecewise model fit the data relatively well, but not ideal
  - $\chi^2(df = 12) = 57.23,\ p < .001,\ RMSEA = .078$
  - borderline model fit, but misfit primarily due to large sample size and high R^2 values of repeated measures (ranges from .88 to .93)
- The linear model is nested within the piecewise model, and piecewise was a drastic improvement in fit over the linear
  - recall linear was $\chi^2(df = 16) = 544.3,\ p < .0001$
- Quadratic not nested in piecewise, so no LRT available
  - but can use Bayesian Information Criterion (BIC) to compare non-nested models; smaller BIC values imply superior fit
  - quadratic BIC=4524.6, piecewise BIC=4449.7; piecewise superior to quadratic

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Bollen, K. A., Harden, J. J., Ray, S., & Zavisca, J. (2014). BIC and Alternative Bayesian Information Criteria in the Selection of Structural Equation Models. *Structural Equation Modeling: A multidisciplinary Journal, 21*, 1–19. https://doi.org/10.1080/10705511.2014.856691

Piecewise Growth Model

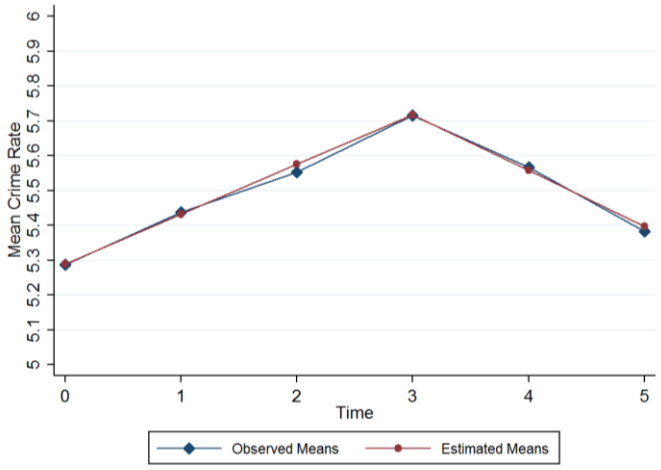
- All estimates are significant and covariances given as correlations



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Piecewise Growth Model

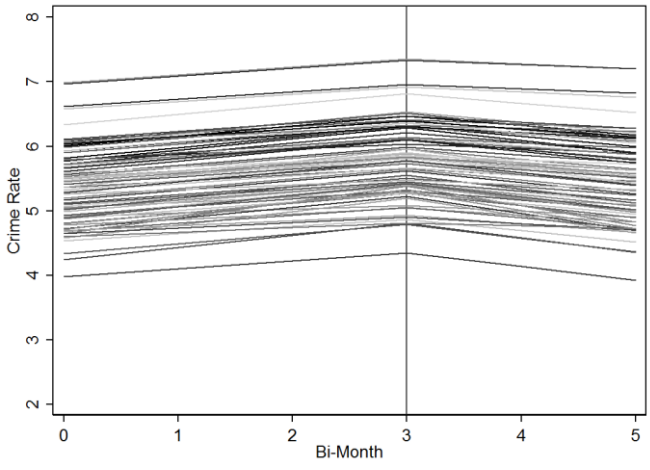
- Can again examine estimated and observed means
  - can observe much improvement in fit (with respect to means)



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Piecewise Growth Model

- We can also examine sampling of first 75 individual trajectories

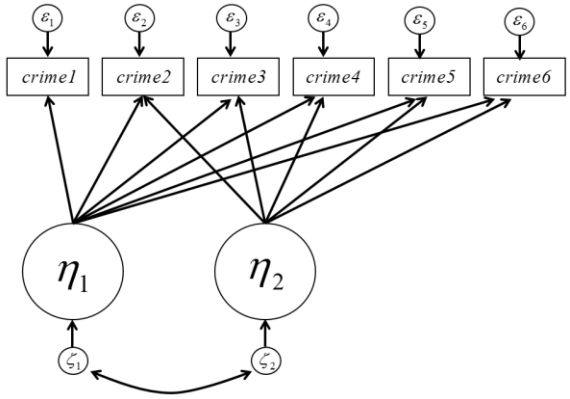


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Much of the variation is simply variation in level, though some differences in rates of change are also apparent.

Freed Loading Growth Model

- A last option is to consider a freed loading model
  - although we don't expect to work well because non-monotonic means



$$\Lambda_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & \lambda_{32} \\ 1 & \lambda_{42} \\ 1 & \lambda_{52} \\ 1 & \lambda_{62} \end{bmatrix}$$

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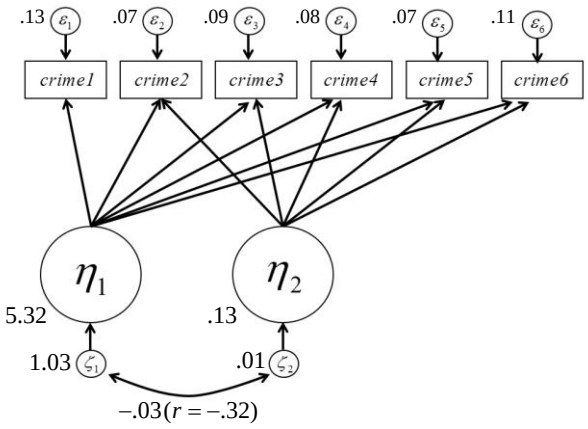
Freed Loading Growth Model

- Freed loading fit better than linear, but worse than piecewise
  - freed loading: $\chi^2(df = 12) = 75.62,\ p < .0001,\ RMSEA = .09$
  - piecewise: $\chi^2(df = 12) = 57.23,\ p < .001,\ RMSEA = .078$
  - quadratic: $\chi^2(df = 12) = 132.15,\ p < .0001,\ RMSEA = .126$
- Freed loading model not nested within piecewise, so no LRT
- But we can again compare BIC of piecewise to freed loading
  - freed loading BIC=4468.1, piecewise BIC=4449.7
  - freed loading model achieves poorer fit compared to piecewise model
- As before, we typically do not interpret estimates because of poor fit, but present to highlight interpretation

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Freed Loading Growth Model

- The estimated factor loadings and latent variable estimates are:



$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2.08 \\ 1 & 3.19 \\ 1 & 2.09 \\ 1 & .48 \end{bmatrix}$$

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Summary

- Time-specific means of UCR measures nonlinear trend over time
- Both linear and quadratic LCM fit extremely poorly
- Freed loading model fit better than quadratic, but itself fit poorly
  - and has odd interpretation given non-monotonic pattern of means
- The piecewise linear model obtained the optimal model fit with significant means and variances in all three latent curve factors
  - significant variability in starting point, initial rate of increase, and subsequent rate of decrease
  - part of poor fit of all models due to high item-level communalities
- Opens possibility of incorporating predictors of change over time

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3.3 Frequently Asked Questions

Objectives

- Questions to be addressed here:
  - Can you estimate higher-order polynomials?
  - How should you choose the knot point for the piecewise?
  - Can you fit a truly nonlinear growth function (e.g., Gompertz) in LCM?
  - What is the minimum number of time points needed to model nonlinearity?
  - Is over-fitting the data a danger with nonlinear LCMs?

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Can You Estimate Higher-Order Polynomials?

- Have seen how to specify 1st and 2nd order polynomial models (linear and quadratic, respectively)
- However, polynomials can in principle go to the p^{th} degree
  - e.g., a third- (cubic), fourth- (quartic), or fifth-degree (quintic) polynomial, etc.
- Each additional degree requires time to be powered accordingly
  - quadratic includes *time* and $time^2$ to define linear and quadratic factors
  - cubic adds $time^3$ to define a cubic factor
  - Additional power of time to define new latent factor for each additional degree
- Higher-degree polynomial models can capture complex trajectory shapes but can be challenging to estimate and interpret, and may overfit the data

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Choosing the Knot Point

- When using a piecewise linear model, a common question is how to choose the knot point defining the pieces
- Sometimes this will be determined a priori based on a known transition point in the data (e.g., school year versus summer break)
- Other times it may be based on inspection of plots of individual scores or means over time (look for sharp elbows)
- If ambiguous, can try several positions for knot point and compare model likelihoods or BICs
- More complex, but can potentially estimate the knot point and even allow it to vary over people (see Kohli, Harring & Hancock, 2013)

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Cudeck, R, & Klebe, K J (2002). Multiphase mixed-effects models for repeated measures data. *Psychological Methods, 7,* 41–63.

Feng, Y, Hancock, G R, & Harring, J R (2019). Latent growth models with floors, ceilings, and random knots. *Multivariate Behavioral Research, 54*, 751–770.

Harring, J. R., Strazzeri, M. M., & Blozis, S. A. (2021). Piecewise latent growth models: beyond modeling linear-linear processes. *Behavior Research Methods*, *53*, 593-608.

Can We Fit Truly Nonlinear Growth Functions?

- The models presented in this chapter all capture nonlinear trajectories yet all are nevertheless *linear in their parameters*
  - Linear & Latent Basis: $y_{ti} = \eta_{1i} + \lambda_t \eta_{2i} + \varepsilon_{ti}$

    (differ in constraining λ_t to linearly increasing time scores vs. estimating)
  - Quadratic & Piecewise: $y_{ti} = \eta_{1i} + \lambda_{2t} \eta_{2i} + \lambda_{3t} \eta_{3i} + \varepsilon_{ti}$

    (differ in how λ_{2t} and λ_{3t} values are defined)
- Notice that each model is expressed as an additive linear combination of the growth factors weighted by (fixed or estimated) factor loadings
- The SEM is designed for models that are linear in their parameters, so these functions are easy to specify and estimate

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Can We Fit Truly Nonlinear Growth Functions?

- Some growth functions, however, are *nonlinear in their parameters*
- For example, we might hypothesize negatively accelerated exponential change over time:

$$y_{ti} = \eta_{1i} + \eta_{1i}\left(1 - e^{\eta_{3i} time_t}\right) + \varepsilon_{ti}$$

- Notice that η_{3i} (rate of growth) enters within an *exponential function*, making this function nonlinear in its parameters
- Several options exist for estimating LCMs with functions like these but these are more complex and seldom used in practice
  - See Grimm, Ram & Hamagami (2011) for an overview
- More common to see LCMs using the functions covered in this chapter that *are* linear in their parameters, given suitability and ease of use

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Blozis, S A (2007). On fitting non-linear latent curve models to multiple variables measured longitudinally. *Structural Equation Modeling, 14*, 179–2001.

Blozis, S A, & Harring, J R (2016). On the estimation of nonlinear mixed-effects models and latent curve models for longitudinal data. *Structural Equation Modeling, 23*, 904–920.

Browne, M W (1993). Structured latent curve models. In C M Cuadras, & R Rao (Eds.) Multivariate analysis: future directions 2 (pp. 171–197). Amsterdam: Elsevier Science.

Grimm, K.J., Ram, N. & Hamagami, F. (2011). Nonlinear growth curves in developmental research. *Child Development, 82*, 1357-1371.

What is the Minimum Number of Time Points Needed?

- The minimum number depends on the complexity of the function
  - For polynomial of degree p, need $p + 2$ or more timepoints (e.g., linear needs three time points, quadratic needs four)
  - For piecewise model with p pieces, need $3 + 2(p - 1)$ or more timepoints (e.g., two-piece needs five time points)
  - Latent basis model needs 4 time points or more
- However, what makes this a bit tricky is that not all individuals must have this number of timepoints
- With both missing data and cohort sequential designs, for instance, it is possible that *no one person* has *T*=4 yet a quadratic function is still identified given at least *T*=4 unique ages across entire sample

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Bollen, K.A., & Curran, P.J. (2006). *Latent Curve Models: A Structural Equation Approach*. John Wiley & Sons: New Jersey.

What is the Minimum Number of Time Points Needed?

- At the extreme, quite complex nonlinear functions can be estimated even if every individual only has two repeated measures
  - e.g., ages 6 through 14 are represented in the data, but all subjects only had *T*=2
- However, we might have little confidence in model results given we are extrapolating from limited information to complex nonlinear functions
- Often, estimation issues will also be encountered, given limited information with which to estimate growth factors
  - especially variances of higher-order factors
- Care is needed in determining if sufficient repeated measures exist to support the estimation of the nonlinear function at hand

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The number of time points needed to identify the model is a separate issue from that of statistical power, which can also be increased with more repeated measures. See

Fan, X. (2003). Power of latent growth modeling for detecting group differences in linear growth trajectory parameters. *Structural Equation Modeling, 10*, 380-400.

Muthén, B. O., & Curran, P. J. (1997). General longitudinal modeling of individual differences in experimental designs: A latent variable framework for analysis and power estimation. *Psychological methods*, 2, 371.

Is Model Over-Fitting a Danger?

- Model over-fitting refers to adding more and more parameters in order to achieve improvements in model fit
  - at the extreme, a saturated model will fit the observed data perfectly
- Moving from intercept-only to linear to curvilinear adds parameters, improving the ability of the model to reproduce the sample data
- The challenge is to balance model parsimony and model fit
  - find model that best represents characteristics of the population without over-fitting to chance features of the sample
- More complex growth functions sometimes introduce unnecessary parameters that undermine parsimony
- This is why careful model comparisons are critical when determining the optimal model

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Summary

- It is possible to estimate a polynomial of any higher degree, assuming sufficient sample size and number of repeated measures are available
- Because of the assumption that the SEM is linear in the parameters, truly nonlinear functions are not easily incorporated into the LCM
  - Options exist for doing so, but these get a bit complex and are rarely used in practice
- Although the minimum number of unique time points across individuals can be determined for a given functional form, this number need not be obtained from each (or possibly even any) individual in the sample
- Like any SEM, care is needed to determine that expanding a model to include additional parameters is resulting in a meaningful improvement in model fit

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Chapter Summary

- Linear LCM implies equal change with respect to time
- Nonlinear LCM implies unequal change with respect to time
- Many ways to estimate nonlinearity
  - we examined quadratic, piecewise, and freed loading (latent basis), but other options exist
- Once optimal unconditional LCM identified, can expand to include one or more time-invariant or time-varying covariates

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Chapter 4
Conditional Latent Curve Model

4.1 Time-Invariant Covariates 4-3
4.2 Demonstration: Conditional LCM for Antisocial Behavior 4-15
4.3 Frequently Asked Questions 4-24

4.1 Time-Invariant Covariates

Objectives

- Define what is a time-invariant covariate, or TIC
- Highlight how TICs can be used to predict latent growth factors
- Explicate the need to probe and plot predictors of slope

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The Unconditional LCM

- Thus far we have focused on *unconditional* LCMs
  - "unconditional" because no predictors in model beyond latent factors
- Goal has been to identify optimal functional form of growth
  - must identify proper fitting unconditional model before expanding
- If variability found in latent growth factors, can consider expanding model to include one or more predictors
  - if there is individual variability in starting point and rate of change, logical question is what between-person characteristics predict growth
- We begin by considering time-invariant covariates, or TICs
- Focus on linear model, but techniques generalize to nonlinear functions

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The Unconditional LCM

- Recall that the *unconditional* linear LCM is defined as

$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i = \begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{Ti} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & T-1 \end{bmatrix} \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \vdots \\ \varepsilon_{Ti} \end{bmatrix}$$

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i = \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} \zeta_{1i} \\ \zeta_{2i} \end{bmatrix}$$

  - "unconditional" because no predictors of $\boldsymbol{\eta}_i$

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Time-Invariant Covariates (TICs)

- TICs person-specific characteristics where value is constant over time
- Examples include biological sex, country of origin, DNA
  - could assess TICs at any time during study and would take on same value
- Can sometimes consider a one-time assessment of a construct as a TIC even though it could potentially change over time
  - e.g., marital status, family income, number of children, etc.
  - care must be taken here because could draw erroneous conclusions about influence of TIC if construct truly changing over time
- TICs incorporated in LCM as predictors of latent growth factors
  - shift in values of TICs shift conditional means of latent factors

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Note that although the term “time-invariant covariate” is commonplace, this sometimes brings about some confusion. In older literatures, a *covariate* is often described as a continuous variable for which one wishes to statistically control, rather than a predictor of primary interest (e.g., in analysis of covariance). Here, the term covariate is used more generally to mean any variable (continuous or categorical) that covaries with the underlying growth process for the repeated measures, whether conceptualized as a predictor of primary interest or a control variable.

Time-Invariant Covariates

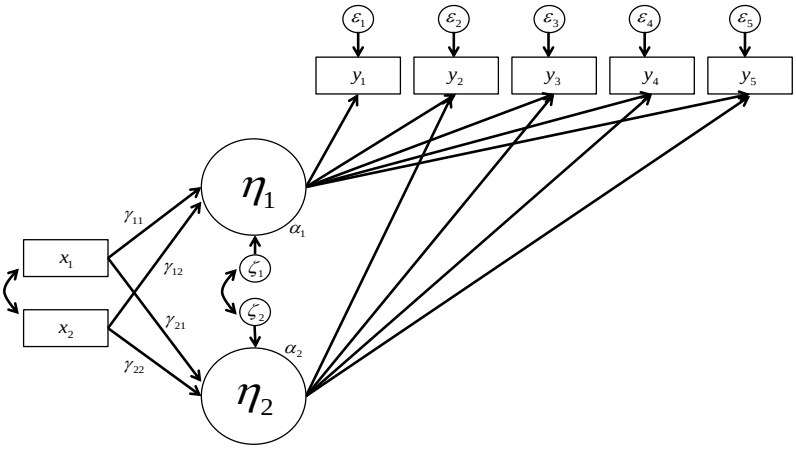
- Inclusion of TICs does not change measurement equation, but predictors entered in structural equation
  - e.g., for two TICs the structural equation is

$$\begin{aligned}\boldsymbol{\eta}_i &= \boldsymbol{\alpha} + \boldsymbol{\Gamma}\mathbf{x}_i + \boldsymbol{\zeta}_i \\ &= \begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} + \begin{bmatrix}\gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22}\end{bmatrix}\begin{bmatrix}x_{1i} \\ x_{2i}\end{bmatrix} + \begin{bmatrix}\zeta_{1i} \\ \zeta_{2i}\end{bmatrix} \\ &= \begin{bmatrix}\alpha_1 + \gamma_{11}x_{1i} + \gamma_{12}x_{2i} + \zeta_{1i} \\ \alpha_2 + \gamma_{21}x_{1i} + \gamma_{22}x_{2i} + \zeta_{2i}\end{bmatrix}\end{aligned}$$

- Really no different than a two-predictor regression equation, but latent intercept and slope are dependent variables

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Time-Invariant Covariates



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In the unconditional LCM the latent factors were characterized by *means* and *variances*. In the presence of one or more TICs, the latent factors are now characterized by *intercepts* and *disturbances*. This change in terminology simply reflects that the latent factors are now conditioned on the predictors.

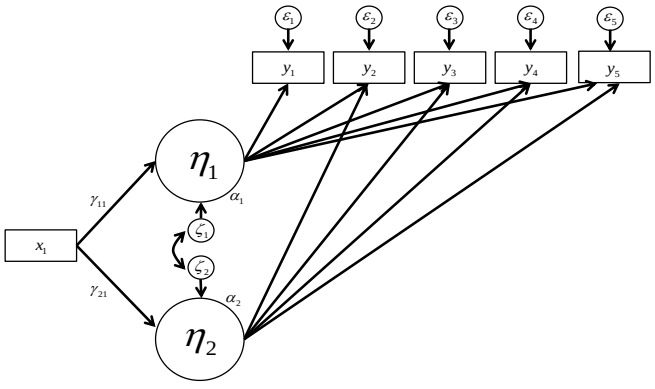
Understanding TIC Influences

- Sometimes challenging to accurately understand joint effects of TICs on set of latent factors
- Interpretation of direct effects of predictors on intercept and slope factor straightforward
  - shift in mean of intercept and slope given corresponding shift in TIC
- But for substantive interpretations we are often interested in *indirect* effect of TICs on RMs as mediated by the latent factors
  - influence of TICs on RMs varies as function of time via slope
  - TIC prediction of slope factor means TIC *differentially* influences RMs
- Can use methods for testing and probing interactions from multiple regression to better understand these effects

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Probing Conditional Effects: 1 Predictor

- Regression of intercept on the TIC reflects a γ_{11}-unit shift in mean of intercept per one-unit shift in the TIC
- Regression of slope on the TIC reflects a γ_{21}-unit shift in mean of slope per one-unit shift in the TIC
- Can probe this effect as a function of time



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Probing Conditional Effects

- The measurement and structural equations for a linear LCM with a single TIC are

$$y_{ti} = \eta_{1i} + \eta_{2i}\lambda_t + \varepsilon_{ti}$$
$$\eta_{1i} = \alpha_1 + \gamma_{11}x_{1i} + \zeta_{1i}$$
$$\eta_{2i} = \alpha_2 + \gamma_{21}x_{1i} + \zeta_{2i}$$

  where main effects of TIC on the growth factors is clearly seen

- However, note that in the reduced form expression, the effect of the TIC directly on the RMs manifests as an *interaction with time*

$$y_{ti} = (\alpha_1 + \alpha_2\lambda_t + \gamma_{11}x_{1i} + \gamma_{21}\lambda_t x_{1i}) + (\zeta_{1i} + \zeta_{2i}\lambda_t + \varepsilon_{ti})$$

- To better understand these effects can define model-implied trajectories at specific values of covariates

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Note that this interaction is precisely that which is obtained in a multilevel model via the cross-level interactions. For details see:

Curran, P.J., Bauer, D.J., & Willoughby, M.T. (2004). Testing main effects and interactions in latent curve analysis. *Psychological Methods, 9,* 220-237.

Probing Conditional Effects

- We begin with the reduced-form expression of the TIC model:

$$y_{ti} = (\alpha_1 + \alpha_2\lambda_t + \gamma_{11}x_{1i} + \gamma_{21}\lambda_t x_{1i}) + (\zeta_{1i} + \zeta_{2i}\lambda_t + \varepsilon_{ti})$$

- We then express this as an expected value conditioned on TIC

$$E(y \mid x_1) = \alpha_1 + \alpha_2\lambda_t + \gamma_{11}x_1 + \gamma_{21}\lambda_t x_1$$

- Finally, we rearrange terms to define intercept and slope conditioned on a specific value of TIC

$$E(y \mid x_1) = (\alpha_1 + \gamma_{11}x_1) + (\alpha_2 + \gamma_{21}x_1)\lambda_t$$

- We can generalize the methods used to probe interactions in regression to better understand these effects

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Probing Conditional Effects

- The model-implied value of outcome at given value of TIC

$$E(y \mid x_1) = \alpha_1 + \alpha_2\lambda_t + \gamma_{11}x_1 + \gamma_{21}\lambda_t x_1$$
$$= (\alpha_1 + \gamma_{11}x_1) + (\alpha_2 + \gamma_{21}x_1)\lambda_t$$

- First parenthetical term is *simple intercept*, second is *simple slope*
  - this is same form as is found in a 2-way interaction in regression
- We'll start with the simple case where the TIC is binary

$$E(y \mid x_1 = 0) = (\alpha_1) + (\alpha_2)\lambda_t$$
$$E(y \mid x_1 = 1) = (\alpha_1 + \gamma_{11}) + (\alpha_2 + \gamma_{21})\lambda_t$$

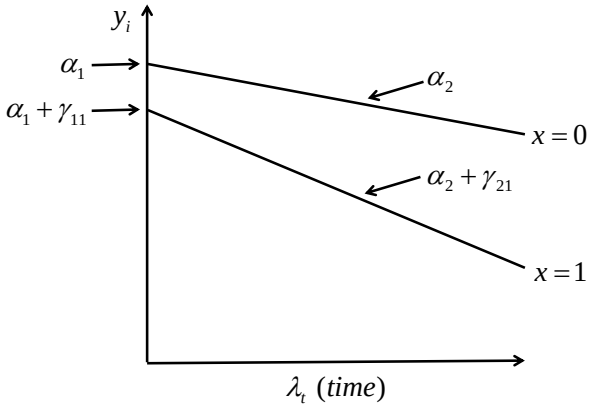
- These relations are easier to see graphically

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It is rather easy to obtain the *point estimates* for the simple intercept and simple slope at different conditional values of the predictor; indeed, we could compute these using a simple calculator. However, it is much more difficult to obtain the corresponding *standard errors* for these point estimates. As such, we'll obtain these by invoking specific commands within our SEM software package.

Probing Conditional Effects

- There are two unique model-implied trajectories, each defined by a *simple-intercept* and a *simple slope* for values 0 and 1 on the TIC



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Probing Conditional Effects

- Straightforward to probe conditional effect for a binary predictor because there are only two possible values: 0 and 1
- More challenging to probe effect of continuous predictor because theoretically an *infinite* number of values to consider
- Must choose a small number of *candidate values* of TIC at which to probe the effect
  - called a *pick-a-point* approach to probing interactions
- Many strategies for picking candidate points are defensible
  - e.g., clinical cut-offs, inter-quartile range
- Here we consider a standard deviation above/below the mean of the TIC

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Probing Conditional Effects

- Only change from before is that instead of having conditional values equal to 0 and 1, we now choose one *sd* below the mean, at the mean, and one *sd* above the mean

$$E(y \mid x_1 = \bar{x} - 1sd_x) = (\alpha_1 + \gamma_{11}[\bar{x} - 1sd_x]) + (\alpha_2 + \gamma_{21}[\bar{x} - 1sd_x])\lambda_t$$
$$E(y \mid x_1 = \bar{x}) = (\alpha_1 + \gamma_{11}\bar{x}) + (\alpha_2 + \gamma_{21}\bar{x})\lambda_t$$
$$E(y \mid x_1 = \bar{x} + 1sd_x) = (\alpha_1 + \gamma_{11}[\bar{x} + 1sd_x]) + (\alpha_2 + \gamma_{21}[\bar{x} + 1sd_x])\lambda_t$$

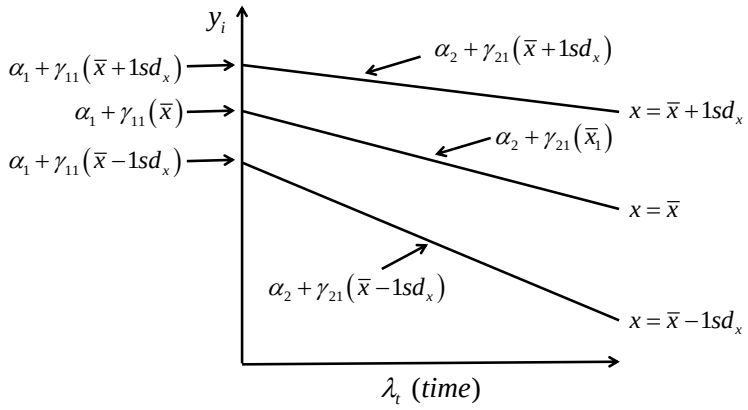
- We can now graph the three conditional relations

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It is common to mean-center predictors to aid in interpretation (although model fit remains unchanged). The mean of a centered variable is equal to zero and the above calculations simplify accordingly. See:

Bauer, D.J., & Curran, P.J. (2005). Probing interactions in fixed and multilevel regression: Inferential and graphical techniques. *Multivariate Behavioral Research, 40*, 373-400.

Probing Conditional Effects



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Testing Conditional Effects

- Rather easy to graph these conditional relations
  - just additive combinations of regression coefficients
- However, primarily interested in tests of these conditional relations
  - is the slope of the trajectory *significantly different from zero* at one standard deviation above the mean of the TIC?
- Can conduct tests by realizing that simple intercept and slope are combinations of other parameter estimates
  - can thus express as compound estimate and compute standard errors
- We will demonstrate one method for conducting these tests, but there are many alternative approaches available

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There are online utilities that will allow for the calculation and plotting of a variety of complex interactions that are not reliant on a single software package (quantpsy.org/interact); for details, see

Preacher, K.J., Curran, P.J., & Bauer, D.J. (2006). Computational tools for probing interactions in multiple linear regression, multilevel modeling, and latent curve analysis. *Journal of Educational and Behavioral Statistics, 31*, 437-448.

Probing When More than One TIC

- Thus far we have only considered a single TIC at a time
  - all models naturally expand to include any number of TICs
- If probe TIC when more than one predictor present, must be aware of meaning of values of other predictors
- If no TICs interact with one another:
  - simple intercepts computed at value of zero on all other predictors
  - simple slopes invariant across other values on all other predictors

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Probing When More than One TIC

- When interpreting simple intercepts:
  - for binary TICs, be aware of what group=0 represents
  - for continuous TICs, value=0 might not be meaningful or even fall within range of observed values
  - a common strategy is to mean-center all continuous TICs so simple intercept is computed at the mean of other TICs (assuming other TICs not involved in interaction with TIC under consideration)

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Interactions among TICs

- If no interactions among TICs, conditioning simple trajectory on other covariates only influences simple intercepts, not slopes
  - more complex if have interactions among two or more TICs because simple slope no longer invariant over other TICs
- For example, could expand LCM to include treatment, biological sex, and treatment-by-sex interaction as predictors of growth
  - tests if magnitude of effect of treatment on intercept and slope significantly different for boys vs. girls
- Can expand probing methods to examine these more complex conditional relations
  - e.g., compute simple trajectories for treatment boys, treatment girls, control boys, and control girls

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Latent Time-Invariant Covariates

- One issue to remember is that, like the regression model, the SEM assumes predictors measured without error
  - Gauss-Markov assumption of predictors being fixed & known
- Similarly, TICs in LCM assumed to be error-free
  - can be obtained in some types of measures (e.g., biological sex), but rarely if ever in other measures (e.g., perceived social support)

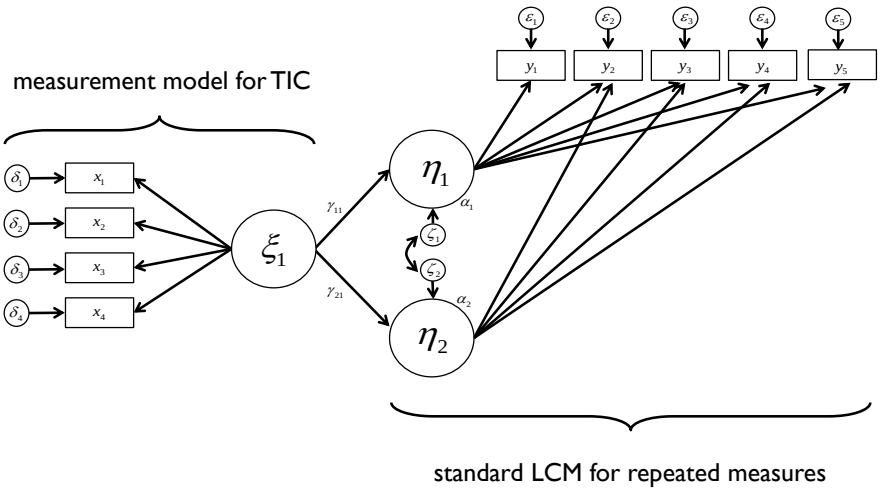
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Latent Time-Invariant Covariates

- Well known that measurement error attenuates sample estimates of regression coefficients
  - on average, sample regression coefficients smaller than population value
- When applicable, can draw on strengths of SEM to estimate latent factors for TICs to model and remove measurement error
  - latent factors "disattenuate" sample estimates of regression coefficients
- Easier to see in path diagram form....

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Latent Time-Invariant Covariates



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For details on latent TICs, see pages 259-262 in:

Bollen, K.A., & Curran, P.J. (2006). *Latent Curve Models: A Structural Equation Approach*. Wiley Series on Probability and Mathematical Statistics. John Wiley & Sons: New Jersey.

Summary

- If variability observed in latent curve factors, can include one or more time-invariant covariates to predict set of factors
- Direct effects on intercept & slope interpreted in the usual way
- But indirect effects of TICs on RMs as mediated by the intercept and slope factors are more complex
- Can compute simple trajectories at specific values of predictor
  - e.g., trajectory for boys vs. girls, for high vs. low aggression, etc.
  - easy to compute point estimates: simply add parameters
  - harder to compute standard errors: must use computer
- Plots also highly beneficial to fully explicate conditional effects

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4.2 Demonstration: Conditional LCM for Antisocial Behavior

Objectives

- Return to the data that assessed mother report of child antisocial behavior between ages 6 and 13
- Review final unconditional LCM
- Expand to include three time-invariant covariates
  - biological sex, cognitive support, emotional support
- Will test, probe, and plot exemplar relations

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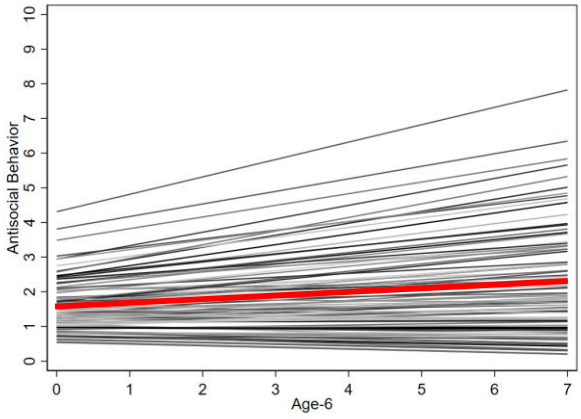
Unconditional Linear Trajectory Model

- Our best fitting model for antisocial behavior from Chapter 2 was a linear latent curve model with homoscedastic residuals
  - $\chi^2(df = 35) = 50.07,\ p = .048,\ RMSEA = .03, CFI = .95, TLI = .97$
  - significant mean intercept (1.6) and slope (.09)
  - significant variability in intercept and slope
  - non-significant covariance between intercept and slope

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Trajectories of Antisocial Behavior

- Found significant variability across children in both initial levels of antisocial behavior at 6 years of age and rate of change over time
- Suggests model could support one or more predictors of growth trajectories
  - Can we now explain this variability?



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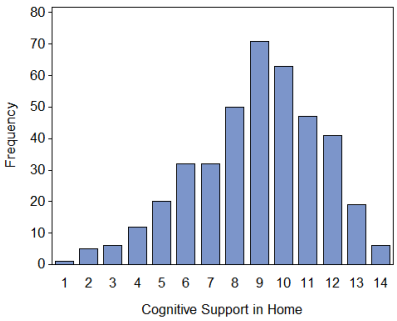
Time-Invariant Covariates

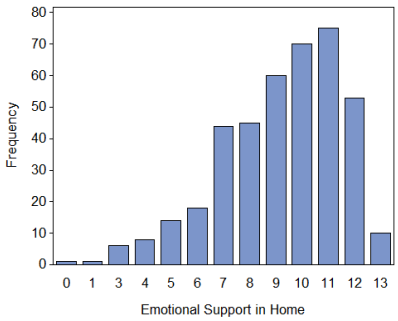
- Time-invariant covariates measured at first assessment:
  - biological sex (**male**): scored 1 if male, 0 if female
  - cognitive support (**homecog**): sum of 14 binary items
    - How often does child read for enjoyment?
    - How often has a family member taken child to any type of museum?
  - emotional support (**homeemo**): sum of 13 binary items
    - How many times in the past week have you had to spank child?
    - Interviewer: Mother's voice conveyed positive feeling about child?

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Descriptive Statistics

Variable	*N*	*Mean*	*Std Dev*	*Minimum*	*Maximum*
male	405	0.50	0.50	0	1
homecog	405	8.89	2.58	1	14
homeemo	405	9.20	2.31	0	13



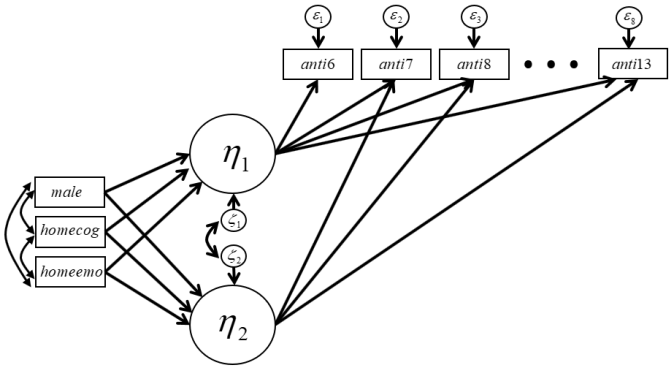


- **homecog** and **homeemo** mean-centered prior to fitting models

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Conditional Linear LCM

- The path diagram for our proposed model is:



- The model fits quite well: $\chi^2(df = 53) = 77.88,\ p = .015,\ RMSEA = .034$

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Conditional Linear LCM

Predictor	Intercept	Slope
Intercept	1.193, p<.001	.078, p=.005
MALE	.817, p<.001	.015, p=.69
HOMECOG	−.025, p=.45	−.017, p=.028
HOMEEMO	−.17, p<.001	.008, p=.41
Multiple R^2	.34	.07

- Females and children from a home with higher levels of emotional support report *lower initial levels* of antisocial behavior
- Children from a home with higher levels of cognitive support report *less steep* increases in antisocial behavior over time
- We can probe these effects to better understand the relations

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Note that there is a much higher proportion of variance explained in the intercept factor relative to the slope factor. This is a common finding in many applications given that individual variability in slopes is often challenging to systematically predict. Also, we typically only formally probe predictors of slope given that this relation is manifested as an interaction with time. However, we can use these same techniques to compute model-implied trajectories for predictors of intercepts only. See, e.g.,

Bauer, D.J., & Curran, P.J. (2005). Probing interactions in fixed and multilevel regression: Inferential and graphical techniques. *Multivariate Behavioral Research, 40*, 373-400.

Curran, P.J., Bauer, D.J., & Willoughby, M.T. (2004). Testing and probing main effects and interactions in latent curve analysis. *Psychological Methods, 9*, 220-237.

Preacher, K.J., Curran, P.J., & Bauer, D.J. (2006). Computational tools for probing interactions in multiple linear regression, multilevel modeling, and latent curve analysis. *Journal of Educational and Behavioral Statistics, 31*, 437-448.

Probing The Effect of Biological Sex

- First compute model-implied trajectories for boys and girls

Sex	Intercept	Slope
Female	1.193, p<.001	.078, p<.001
Male	2.01, p<.001	.094, p<.001

- Boys are starting at higher initial levels of antisocial behavior but both groups are increasing at a similar rates over time
  - know it is higher initial levels for boys because sex significantly predicted intercept factor
  - know it is similar rates of change for boys and girls because sex did not significantly predict slope
- It is easier to see these relations in a plot

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To compute the sex-specific trajectories, we use the methods for probing and plotting described in the prior section. More specifically, recall that the model-implied trajectories within each of two groups is define as:

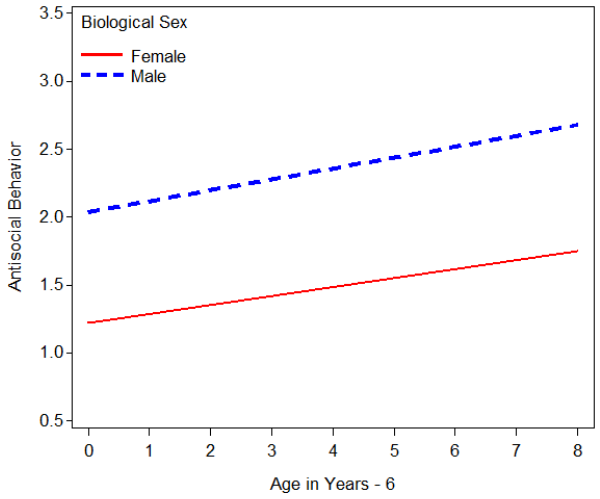
$$E(y \mid x_1 = 0) = (\alpha_1) + (\alpha_2)\lambda_t$$
$$E(y \mid x_1 = 1) = (\alpha_1 + \gamma_{11}) + (\alpha_2 + \gamma_{21})\lambda_t$$

As we described earlier, these are computed while holding all other predictors at a value of zero (or "controlling" for other predictors). Because HOMECOG and HOMEEMO are mean centered (and thus have a sample mean equal to zero), the conditional values for boys are girls are computed at the mean of HOMECOG and HOMEEMO. For our example, the sample values are

$$E(anti \mid male = 0) = (1.193) + (.078)\lambda_t$$
$$E(anti \mid male = 1) = (1.193 + .817) + (.078 + .015)\lambda_t = (2.01) + (.094)\lambda_t$$

Although it is straightforward to compute the point estimates for the sex-specific trajectories, we must use a software package to compute the associated standard errors. We show how this is done in the software demonstration notes.

Simple Trajectories by Sex



- Clear difference in simple intercepts
  - reflected in significant regression of the intercept factor on **male**
- But little difference in simple slopes
  - reflected in non-significant regression of the slope factor on **male**
- Thus boys and girls differ in *level* but follow roughly *parallel* trajectories

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We know that the slopes do not significantly differ between boys and girls because MALE did not predict the slope factor; it only predicted the intercept factor.

Probing The Effect of HOMECOG

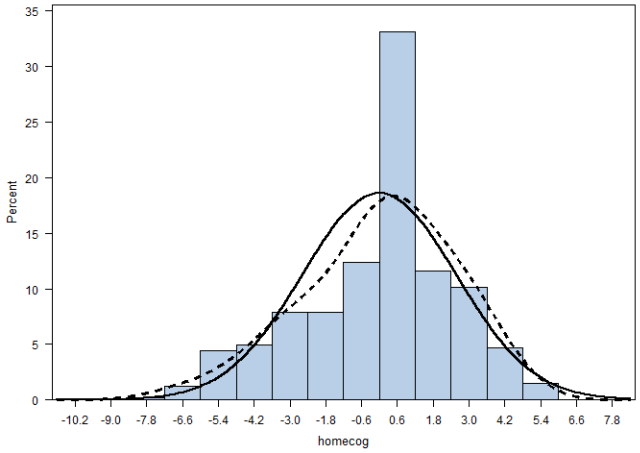
- Will probe effects of HOMECOG because significant predictor of slope
  - must compute conditional trajectories to understand nature of effect
- Probing a binary predictor is straightforward: only two possible values
- Probing a continuous predictor is more challenging
  - in principle, there are an infinite number of possible conditional values
- Many strategies available for selecting conditional values
- Select one *sd* below mean, at the mean, and one *sd* above mean
  - because mean centered, the sample mean equals zero
  - sample standard deviation equals 2.57

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We probe HOMECOG as this is the only significant predictor of the slope factor.

Distribution of Mean-Centered HOMECOG

- Have centered HOMECOG
  - mean=0
  - standard deviation=2.57
  - no assumption about the *shape* of distribution because it is a predictor



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We show this distribution to highlight that there are observed cases at plus and minus one standard deviation around the mean. This is not always the case -- be sure in your own data that there are observed values at whatever conditional value at which you choose to probe the effect.

Probing The Effect of HOMECOG

- Simple slopes at low, medium and high cognitive support are:

HOMECOG	Intercept	Slope
Below the Mean	1.257, p<.001	.123, p<.001
At the Mean	1.193, p<.001	.078, p=.005
Above the Mean	1.129, p<.001	.033, p=.334

- Higher initial values of cognitive support are associated with systematically *lower initial levels* of antisocial behavior
- Higher initial values of cognitive support are associated with increasingly *less steep* slopes of antisocial behavior
  - indeed, slope of antisocial behavior is not significantly different from zero for children one *sd* above the mean of cognitive support

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We will use the same methods to compute the model-implied trajectories as we did for our binary measure of biological sex, except here we have three conditional values: one *sd* below the mean, at the mean, and one *sd* above the mean. Because HOMECOG is mean centered, the sample mean equals zero.

In the prior section we showed that the simple trajectories are defined as:

$$E\left(y \mid x_1 = \bar{x} - 1sd_x\right) = \left(\alpha_1 + \gamma_{11}\left[\bar{x} - 1sd_x\right]\right) + \left(\alpha_2 + \gamma_{21}\left[\bar{x} - 1sd_x\right]\right)\lambda_t$$

$$E\left(y \mid x_1 = \bar{x}\right) = \left(\alpha_1 + \gamma_{11}\bar{x}\right) + \left(\alpha_2 + \gamma_{21}\bar{x}\right)\lambda_t$$

$$E\left(y \mid x_1 = \bar{x} + 1sd_x\right) = \left(\alpha_1 + \gamma_{11}\left[\bar{x} + 1sd_x\right]\right) + \left(\alpha_2 + \gamma_{21}\left[\bar{x} + 1sd_x\right]\right)\lambda_t$$

Our sample specific values are

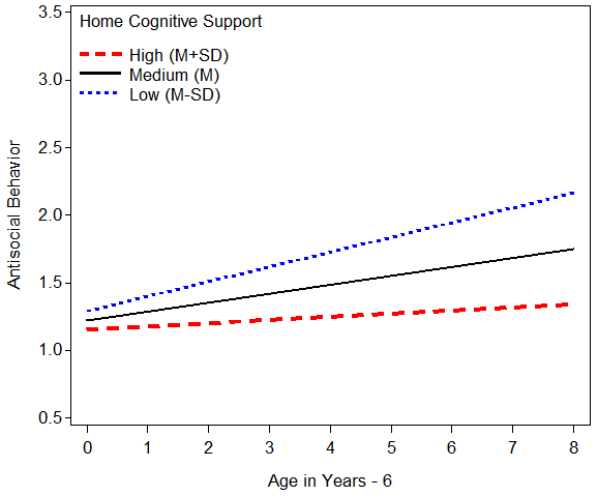
$$E\left(anti \mid cog = low\right) = \left(1.193 + \left[-.025 \times -2.57\right]\right) + \left(.078 + \left[-.017 \times -2.57\right]\right)\lambda_t = 1.257 + .123\lambda_t$$

$$E\left(anti \mid cog = med\right) = \left(1.193 + \left[-.025 \times 0\right]\right) + \left(.078 + \left[-.017 \times 0\right]\right)\lambda_t = 1.193 + .078\lambda_t$$

$$E\left(anti \mid cog = high\right) = \left(1.193 + \left[-.025 \times 2.57\right]\right) + \left(.078 + \left[-.017 \times -2.57\right]\right)\lambda_t = 1.129 + .033\lambda_t$$

Notice that, like earlier, these point estimates are computed for values of zero on all other predictors. In this example, that means that these are the trajectories for girls at the mean of HOMEEMO. Also, as with the conditional trajectories computed for boys and girls, we must obtain standard errors for the conditional intercepts and slopes using a software package. We show how these are obtained in the software demonstration notes.

Simple Trajectories by Cognitive Support



- Little difference in simple intercepts
  - reflected in non-significant effect of **homecog** on intercept factor
- But simple slope clearly *decreases* with *increasing* cognitive support
  - reflected in significant prediction of latent slope factor
- Importantly, slope at high level of **homecog** does not significantly different from zero

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Summary

- Unconditional LCM showed significant variability in intercept and slope factors of antisocial behavior
- Conditional LCM showed the joint contributions of biological sex, cognitive support, and emotional support
- Girls and children from homes with higher emotional support report lower initial levels of antisocial behavior
- Children from homes with higher cognitive support report less steep increases in antisocial behavior over time
  - trajectories not significantly different from zero at highest levels of cognitive support

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4.3 Frequently Asked Questions

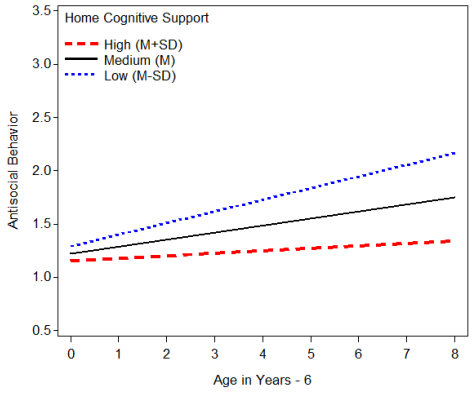
Objectives

- Questions addressed here:
  - Can you test whether two simple slopes are different from each other?
  - What if none of the simple slopes are significant?
  - How do you plot and probe TIC effects on nonlinear trajectories?
  - Can regions-of-significance be used in the LCM?
  - Can growth factors be used as predictors of other distal outcomes?

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Can You Test if Two Simple Slopes are Different?

- We typically test each simple slope against a null of zero
  - Here slope significantly different from zero at High and Medium but not low HOMECOG
- Sometimes people wonder if they can test one against another
  - Slope for High ≠ Slope for Medium?
- We already have this test! This is the null hypothesis that the test of the interaction (prediction of slope factor) evaluates
  - Significant prediction of slope factor implies slopes differ over values of TIC



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Aiken, L. S., West, S. G., & Reno, R. R. (1991). Multiple regression: Testing and interpreting interactions. sage.

What if no Simple Slopes are Significant?

- Just as occurs in the traditional multiple regression model, it is possible to obtain a significant interaction yet none of the resulting simple slope estimates are significantly differ from zero
- Although counter-intuitive, this finding is fully explicable
- Remember a test of an interaction is a test of *parallelism*
  - that is, do the simple slopes meaningfully deviate from being parallel
- It is possible that one simple slope is slightly positive, one is flat, and one is slightly negative
  - although the slopes differ from *one another,* no single slope differs from *zero*
- This is why post-hoc probing and plotting of simple slopes is critical

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Conditional Nonlinear Trajectory Models

- Here explicated techniques for plotting and probing using a conditional **linear** trajectory model
- However, procedures are equally applicable with quadratic, piecewise, or freed loadings model described in Chapter 3
  - Quadratic: Compute, test, and plot simple intercept and simple linear and quadratic trends at selected values of TIC
  - Piecewise: Compute, test, and plot simple intercept and simple slope for each piece at selected values of TIC
  - Freed loadings: Compute and test simple intercept and "shape factor" values at selected values of TIC. To plot implied trajectory, multiply through estimated factor loadings

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Grimm, K J, Ram, N, & Hamagami, F (2011). Nonlinear curves in developmental research. *Child Development, 82*, 1357–1371.

Meredith, W, & Tisak, J (1990). Latent curve analysis. *Psychometrika, 55*, 107–122.

Can Regions of Significance be Computed?

- To probe the interaction with time, we used the *pick-a-point* procedure
  - e.g., we select candidate values of the moderator and determine if the resulting simple slope significantly differs from zero
- However, it is possible to flip this problem on its head
  - instead of testing the significance of a simple slope at a given value of the moderator, we can instead determine the values of the moderator at which the simple slope moves in or out of significance
- Called a *region of significance* (a.k.a. *Johnson-Neyman Technique*)
  - these regions can be graphically displayed in very compelling ways
- See Bauer & Curran (2005), Curran et al. (2004) & Preacher et al. (2006)

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These citations are presented on page 4-18 above.

Can Growth Factors be Used as Predictors?

- Here we focused on TICs as predictors of latent growth factors
- Because the LCM is embedded within the general SEM, it is quite easy to estimate a model in which the growth factors predict a distal outcome
  - e.g., trajectory of alcohol use predicting later substance use disorder diagnosis
- Although easy to estimate, many complexities arise in both the estimation and interpretation of growth-as-predictors models
  - a key challenge is the regression coefficients can change radically under alternative time coding schemes
- Great care is needed in the use of these type of LCMs
  - see McCormick, Curran & Hancock (in press) for complete details

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McCormick, E.M., Curran, P.J., and Hancock, G.R. (in press). Latent growth factors as predictors of distal outcomes. *Psychological Methods.*

Summary

- The test of the interaction (prediction of slope factor) tells you whether any two simple slopes differ from one another
- It is possible to obtain a significant prediction of slope yet none of the simple trajectories differ from zero
- Probing / plotting procedures extend easily to nonlinear models
- Methods are well developed and readily available to compute and plot regions of significance to further explicate predictions of the slope factor
- Growth factors can be used as predictors of distal outcomes, but this is a dangerous endeavor requiring great care in both estimation and interpretation

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Chapter Summary

- Once optimal unconditional LCM identified (linear, nonlinear, etc.), can expand to include one or more time-invariant covariates
- Effects of predictors on latent growth factors can be probed to better understand the conditional nature of these relations
- Can test significance of simple intercepts and slopes and plot model-implied trajectories at specific values of the predictor
- Can expand model to include interactions among predictors themselves

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Chapter 5
Time-Varying Covariates and Multivariate Latent Curve Models

5.1 Time-Varying Covariates 5-3

5.2 Demonstration: Self-Efficacy & School Grades, Part 1 5-14

5.3 The Multivariate Latent Curve Model 5-19

5.4 Demonstration: Self-Efficacy & School Grades, Part 2 5-28

5.5 Frequently Asked Questions 5-32

5.1 Time-Varying Covariates

Objectives

- Differentiate time-varying versus time-invariant covariate
- Expand measurement equation to include one or more TVCs
- Expand TVC model to include TICs

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Time-Invariant Covariates, Redux

- Earlier we showed how the unconditional LCM can be expanded to include one or more time-invariant covariates
  - 1-unit change in TIC → γ-unit shift in conditional mean of growth factor
- Key assumptions:
  - numerical value of TICs independent of time
  - effect of TIC on outcome entirely mediated by latent growth factors
- But what if values of covariate vary with respect to time?
  - numerical values not independent of time?
  - effects of covariate on outcome not entirely mediated by latent factors?
- Can expand LCM to include time-varying covariates

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Time-Varying Covariates

- Time-varying covariates (TVCs) are variables that are allowed to obtain unique numerical values across time
  - e.g., mood, change in status like marriage or arrest, substance use, etc.
- Can conceptualize TVC as traditional covariate that adjusts outcome as part of growth process
  - e.g., growth model of reading ability adjusting for days of instruction
- Or can conceptualize TVC as fundamental aspect of full model
  - e.g., trajectories of substance use in part determined by direct influences of time-specific measures of stress
- Helpful to think about TVCs in terms of time-specific residuals

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Unconditional LCM

- Recall scalar equation for an unconditional linear LCM is

$$y_{ti} = \eta_{1i} + \lambda_t \eta_{2i} + \varepsilon_{ti}$$

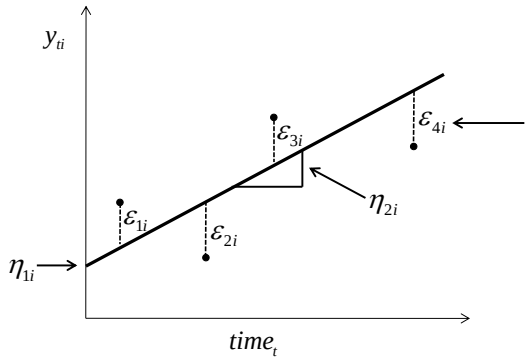
- Up to now, have assumed residual (ε_{ti}) is random error
  - i.e., any deviation of observed value from the underlying trajectory is due to sampling variability
- But part of residual could be due to one or more omitted time-specific influences that serves to offset measure from trajectory
  - e.g., parents get divorced that year; a new medication is prescribed
- Easier to see this graphically

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Unconditional LCM

- Goal is to bring in time-specific predictor to try to partially explain offset of observed measure from underlying trajectory

$$y_{ti} = \eta_{1i} + time_t \eta_{2i} + \varepsilon_{ti}$$



Part of time-specific deviation may be random variability, but part may be due to some time-specific influence

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Conceptually we can think of a TVC as exerting a time-specific "shock" that partially offsets the outcome relative to the underlying trajectory.

Time-Invariant Covariates

- In conditional TIC model the measurement equation is same as unconditional model, but structural equation incorporates TICs:

$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \underbrace{\boldsymbol{\Gamma}\mathbf{x}_i}_{\text{TICs enter model here}} + \boldsymbol{\zeta}_i$$

- Have opposite situation with the unconditional TVC model
  - structural equation will remain the same as the unconditional LCM
  - measurement equation expands to include time-specific effects of TVCs

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Time-Varying Covariates: Unconditional

- To differentiate TVCs from TICs, we will refer to TVCs as z_{ti}
  - measure z can take on a unique value at time t for person i
- Because there are multiple values of TVCs for each case, we will organize these in vector, just as we did for repeated measures

$$\mathbf{z}_i = \begin{bmatrix} z_{1i} \\ z_{2i} \\ \vdots \\ z_{Ti} \end{bmatrix}$$

- This vector typically same dimension as repeated measures
  - most often assessment of TVC same time period as outcome, but not always

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Time-Varying Covariates: Unconditional

- Measurement equation expands to include vector of TVCs
- For a single TVC, the measurement equation is

$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \mathbf{K}\mathbf{z}_i + \boldsymbol{\varepsilon}_i$$

- The matrix $\mathbf{K}$ contains the regression coefficients that link the TVCs in $\mathbf{z}_i$ to the repeated measures of the outcome in $\mathbf{y}_i$
- Thus the TVCs directly predict the repeated outcome

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We use the notation of kappa (i.e., $\mathbf{K}$) to differentiate the TVC regressions from the TIC regressions captured in gamma (i.e., $\boldsymbol{\gamma}$). Both $\mathbf{K}$ and $\boldsymbol{\gamma}$ represent regression parameters, but these simply reside in different parts of the model.

Time-Varying Covariates: Unconditional

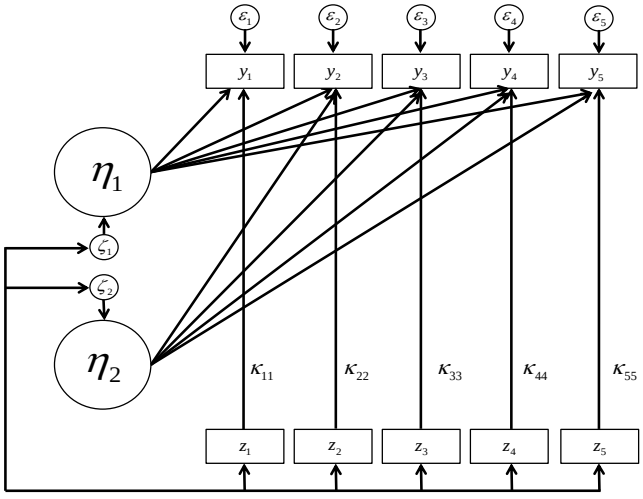
- Because TVCs do not directly influence factors, the structural equation is defined by just factor means and individual deviations

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$$

- However, the latent factors now capture change in the outcome *net* the influence of the TVCs
  - i.e., estimating trajectories of outcomes that have been adjusted for TVC
- This is key to TVC model: the joint inclusion of the latent growth process and the time-specific effects of the TVC

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Time-Varying Covariates: Unconditional



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Time-Varying Covariates: Unconditional

- Elements of measurement equation for T=5 are

$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \mathbf{K}\mathbf{z}_i + \boldsymbol{\varepsilon}_i$$

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \\ y_{4i} \\ y_{5i} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} + \begin{bmatrix} \kappa_{11} & 0 & 0 & 0 & 0 \\ 0 & \kappa_{22} & 0 & 0 & 0 \\ 0 & 0 & \kappa_{33} & 0 & 0 \\ 0 & 0 & 0 & \kappa_{44} & 0 \\ 0 & 0 & 0 & 0 & \kappa_{55} \end{bmatrix} \begin{bmatrix} z_{1i} \\ z_{2i} \\ z_{3i} \\ z_{4i} \\ z_{5i} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \varepsilon_{3i} \\ \varepsilon_{4i} \\ \varepsilon_{5i} \end{bmatrix}$$

- We can expand the matrix operations to express each repeated measure as a joint function of latent curves and TVC

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K is a square matrix and the diagonal represents the contemporaneous (within-time) regression of the outcome on the TVC. Below the diagonal represent lagged effects (earlier measures of the TVC predicting later measures of the outcome) and above the diagonal are impossible effects that should always be set to zero (later measures of the TVC predicting earlier measures of the outcome).

Time-Varying Covariates: Unconditional

$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \mathbf{K}\mathbf{z}_i + \boldsymbol{\varepsilon}_i \longrightarrow \begin{bmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \\ y_{4i} \\ y_{5i} \end{bmatrix} = \begin{bmatrix} \eta_{1i} + \kappa_{11}z_{1i} + \varepsilon_{1i} \\ \eta_{1i} + 1\eta_{2i} + \kappa_{22}z_{2i} + \varepsilon_{2i} \\ \eta_{1i} + 2\eta_{2i} + \kappa_{33}z_{3i} + \varepsilon_{3i} \\ \eta_{1i} + 3\eta_{2i} + \kappa_{44}z_{4i} + \varepsilon_{4i} \\ \eta_{1i} + 4\eta_{2i} + \kappa_{55}z_{5i} + \varepsilon_{5i} \end{bmatrix}$$

- To better see how TVC "adjusts" the outcome in the growth model, we can rearrange terms for just a single time point

$$\left(y_{3i} - \kappa_{33}z_{3i}\right) = \eta_{1i} + 2\eta_{2i} + \varepsilon_{3i}$$

  - simply highlights that growth process is estimated for RMs net TVCs

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The latent factors now capture the trajectories of the outcomes *net* the effects of the TVC; that is, the outcomes are conditioned on the TVCs. Note also that all TVCs should covary with all latent factors; this ensures that the TVCs are uniquely predicting the outcome net the latent factors and vice versa.

Time-Varying Covariates: Unconditional

- Because repeated measures expressed as joint function of growth factors and TVCs, two equivalent interpretations
  1. The growth factors estimate the mean and variance of trajectories of outcome above-and-beyond influence of time-varying covariates
     - e.g., estimate individual trajectories in reading ability after having adjusted for the time-specific effects of number of days of instruction
  2. The time varying covariates estimate the unique prediction of outcome above-and-beyond influence of the growth factors
     - e.g., estimate unique time-specific effects of environmental stress on substance use above-and-beyond the underlying trajectory process
- These are equivalent, so choice depends on theory

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Time-Varying Covariates: Unconditional

- Sequence of model building with TVCs not always clear
- Can build unconditional model first, then expand to TVCs
  - identify optimal functional form, then expand to include joint influences of TVCs
- Can also build growth model in presence of TVCs
  - well known TVCs can influence mean and covariance structure of RMs
    - often whole point of TVC -- to "adjust" RMs for time-specific influences
  - but presence of TVCs can then directly impact functional form
    - unadjusted RMs might be nonlinear but TVC-adjusted RMs might be linear
  - if theory strongly supports TVCs, may need to include at outset

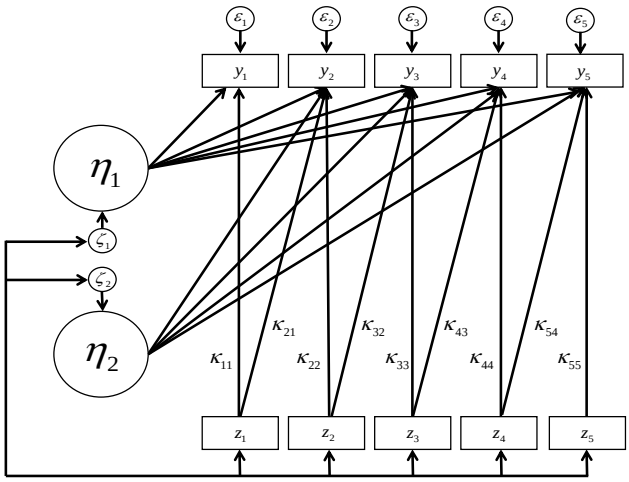
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Contemporaneous vs. Lagged Effects

- Standard TVC model defined with contemporaneous effects
  - TVC at t=1 predicts outcome at t=1, TVC at t=2 predicts outcome at t=2, etc.
  - within-time relations despite longitudinal design
- But TVC regressions can easily be lagged in a variety of ways
  - TVC at t=1 predicts outcome at t=2 ignoring effect of the TVC at t=2
  - TVC at t=1 predicts outcome at t=2 net effect of TVC at t=2
- Can include longer lags if supported by data

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Time-Varying Covariates: Lagged Effects



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The lagged effects simply introduce four new elements into the $\mathbf{K}$ matrix:

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \\ y_{4i} \\ y_{5i} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} + \begin{bmatrix} \kappa_{11} & 0 & 0 & 0 & 0 \\ \kappa_{21} & \kappa_{22} & 0 & 0 & 0 \\ 0 & \kappa_{32} & \kappa_{33} & 0 & 0 \\ 0 & 0 & \kappa_{43} & \kappa_{44} & 0 \\ 0 & 0 & 0 & \kappa_{54} & \kappa_{55} \end{bmatrix} \begin{bmatrix} z_{1i} \\ z_{2i} \\ z_{3i} \\ z_{4i} \\ z_{5i} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \varepsilon_{3i} \\ \varepsilon_{4i} \\ \varepsilon_{5i} \end{bmatrix}$$

Because the LCM with contemporaneous TVC effects is nested within the LCM with lagged TVC effects, we can use an LRT to compare the improvement in model fit with the inclusion of the lagged effects (as well as testing the equality of the TVC effects across time).

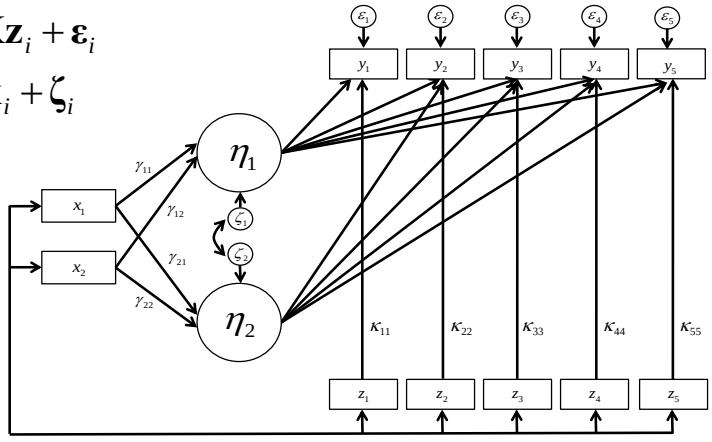
Extensions of Standard TVC Model

- Can test for interactions between effect of TVC and time
- Can include more than one TVC at a time
- Can test interactions among TVCs themselves
- Can define TVCs as latent factors to model measurement error
  - as exogenous predictor, TVCs assumed error free
  - latent factor of TVCs disattenuates estimate of regression parameter
- Finally, can combine TVC influences with one or more TICs

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Conditional TVC Model

$$\mathbf{y}_i = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \mathbf{K}\mathbf{z}_i + \boldsymbol{\varepsilon}_i$$
$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\Gamma}\mathbf{x}_i + \boldsymbol{\zeta}_i$$



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Note the more complex covariance structures. As before, all exogenous predictors covary with one another (that is, the TVCs and TICs), and the endogenous latent factors covary with one another (that is, the intercept and slope), but the factors do not directly covary with the TICs or the TVCs.

Conditional TVC Model

- Simultaneously combines elements of TIC and TVC models:
  - TVCs uniquely predict RMs above-and-beyond underlying latent growth factors
  - Latent growth factors capture growth in RMs adjusted for impact of TVCs
  - Regression of factors on TICs reflects conditional mean shift in trajectories of TVC-adjusted outcomes as function of TICs
- Could examine interactions between TICs and TVCs, but becomes increasingly complex, especially with many RMs and many TICs
  - Later we will describe multiple group model that provides tests of interactions across discrete group membership

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Summary

- TVC model allows for covariates that vary in value across time
- TVC effects can be contemporaneous or lagged
- Magnitude of TVC effect can interact with time
- Multiple TVCs can be included and can interact with one another
- TVCs assumed to be measured error free, but given sufficient data can model as latent factors
- Multiple TICs can be included and can interact with TVCs

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5.2 Demonstration: Self-Efficacy & School Grades, Part 1

Objectives

- Fit a series of time-varying covariate LCMs to data drawn from 964 adolescents
- Ultimate goal is to examine longitudinal bivariate relations between self-efficacy and school grades
- Begin with test of developmental trajectories of self-efficacy in the presence of the time-specific prediction of school grades

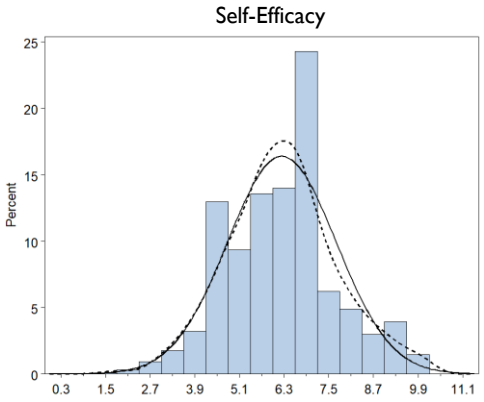
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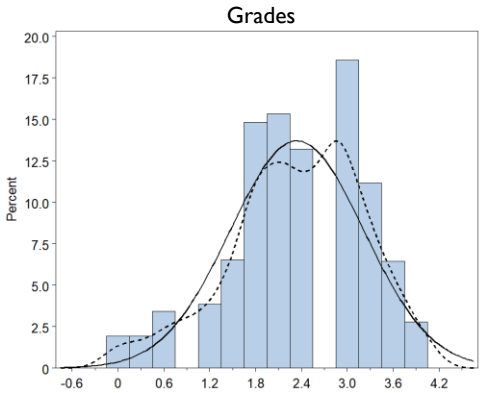
Example: Self-Efficacy & School Grades

- Data collected on 964 high school students in St. Paul public schools
  - 52% of subjects are female; 51% report holding part-time job at time 1
- Annual repeated assessments obtained up to four times
  - modest attrition over time, but will retain all cases for modeling
- Primary outcome variable is self-efficacy (mastery), scored on a range from 0 to 10
- Interest in how self-efficacy changes over the high-school period and whether trajectories of self-efficacy related to grades
- Data courtesy of Jeylan Mortimer, Director of the Life Course Center, University of Minnesota

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Self-Efficacy & Grades at Year 1





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Age-Specific Means and Variances

- The age-specific means and standard deviations for self-efficacy and school grades are:

		Year 1	Year 2	Year 3	Year 4
Self-Efficacy	mean	6.24	6.39	6.53	6.67
	sd	1.46	1.57	1.57	1.53
Grades	mean	2.34	2.44	2.49	2.66
	sd	.87	.85	.84	.83

- Although the standard deviations are approximately equal over time, the means are clearly increasing throughout high school

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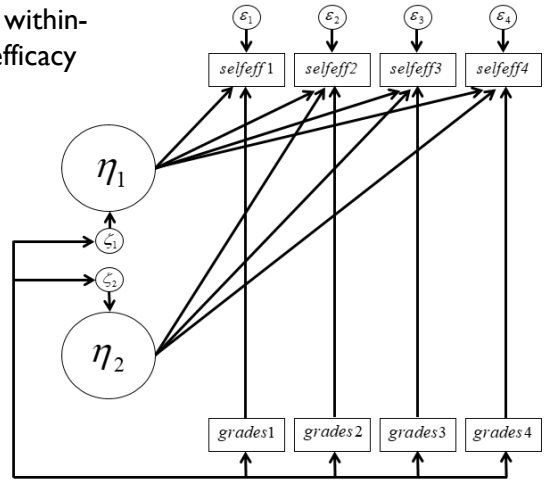
Linear LCM for Self-Efficacy

- We began by fitting a series of LCMs to self-efficacy to find the optimal functional form
  - concluded that linear model with heteroscedastic residuals was optimal
- Model fit the data very well:
  - $\chi^2(df = 5) = 6.09,\ p = .30,\ RMSEA = .015,\ CFI = .99,\ TLI = .99$
- Means and variances of slope and intercept factor all significant
  - model-implied starting point of 6.25
  - model-implied linear rate of change of .14
  - significant variability in both starting point and rate of change
- Next expand model to include school grades as a TVC

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LCM of Self-Efficacy with Grades as TVC

- Expand LCM to include within-time regression of self-efficacy on school grades



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LCM of Self-Efficacy with Grades as TVC

- The model fit the data well:
  - $\chi^2(df = 9) = 13.2,\ p = .15,\ RMSEA = .022,\ CFI = .99,\ TLI = .99$
- Next imposed equality constraints on the set of TVC estimates
  - e.g., self-efficacy1 on grades1 equal to self-efficacy2 on grades 2, etc.
- The resulting model fit well:
  - $\chi^2(df = 12) = 13.74,\ p = .32,\ RMSEA = .012,\ CFI = .99,\ TLI = .99$
- The restriction did not lead to a decrement in fit via the LRT:
  - $\chi^2_{\Delta}(df = 3) = .54,\ p = .92$
- Estimated lagged effect of grades predicting self-efficacy
  - did not lead to improvement in model fit and was not retained

CenterStat

LCM of Self-Efficacy with Grades as TVC

- Final model retained within-time efficacy on grades held equal over time
  - $\chi^2(df = 12) = 13.74,\ p = .32,\ RMSEA = .012,\ CFI = .99,\ TLI = .99$
- TVC effects positive and significant indicating higher grades associated with higher self-efficacy, net trajectory of self-efficacy
  - raw TVC coefficient was .186; standardized TVC coefficient .10
- Means of self-efficacy intercept & slope factors now *net* effects of grades
  - without TVC: intercept and slope were 6.25 & .14, respectively
  - with TVC: intercept and slope were 5.81 and .13, respectively
- Repeated measures of self-efficacy partially explained by joint effects of underlying growth process and time-specific effects of grades

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LCM of Self-Efficacy with Grades as TVC

- These are very interesting results, but there is a major limitation
- Although we are explicitly modeling change over time for self-efficacy, we are not imposing any time-linked structure for grades
  - but earlier we observed means of grades are increasing over time
- In next section we explore how to build a growth model for both self-efficacy and grades simultaneously
  - we will then return to these same data and fit multivariate LCMs

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Summary

- Results indicated that self-efficacy was best characterized by a linear trajectory over time
- TVC effects of grades uniquely and positively predicted self-efficacy above-and-beyond the underlying influences of the latent trajectories of efficacy
- However, all TVC models assume that grades itself is not changing systematically over time
- If TVC itself is growing, must expand LCM to include growth processes for both constructs at once

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5.3 The Multivariate Latent Curve Model

Objectives

- Reparameterize LCM to allow for growth in TVCs
- Simultaneously estimate growth in two processes at once
- Introduce covariances among latent growth factors and residuals
- Rescale some covariances to factor regressions
- Regress all latent factors on TICs

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Limitations of TVC Model

- TVC model allows for covariates that vary in value across time
- TVCs can obtain unique numerical value at each time point, but values assumed to not systematically relate to passage of time
  - i.e., model assumes no growth process for TVCs
- Reasonable assumption if TVC not changing over time
  - e.g., onset of diagnosis, number of days ill, argument with spouse
- But can be significantly limited if TVC itself is growing over time
  - e.g., alcohol use in adolescence, stress with approach of bar exam, etc.
- Can expand LCM to allow for growth in two constructs at once

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Multivariate LCM

- There are many situations in which we have substantive interest in how two constructs change together over time
  - negative affect and substance use
  - perceived stress and psychological wellbeing
  - math skills and reading ability
- Surprisingly straightforward to both conceptualize and estimate a multivariate LCM
  - define two univariate LCMs and link them via the factors and residuals
- We must expand our notation slightly to incorporate a second set of endogenous repeated measures

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Multivariate LCM: Repeated Measures

- Repeated measures are stacked in single vector ordered y then z

$$\mathbf{y}_i = \begin{bmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \\ y_{4i} \\ y_{5i} \end{bmatrix} \qquad \mathbf{z}_i = \begin{bmatrix} z_{1i} \\ z_{2i} \\ z_{3i} \\ z_{4i} \\ z_{5i} \end{bmatrix} \qquad \begin{pmatrix} \mathbf{y}_i \\ \hline \mathbf{z}_i \end{pmatrix} = \begin{bmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \\ y_{4i} \\ y_{5i} \\ \hline z_{1i} \\ z_{2i} \\ z_{3i} \\ z_{4i} \\ z_{5i} \end{bmatrix}$$

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The numbering of elements in the stacked vector is a bit inconsistent but helps aid in interpretation. For example, the first measure of the TVC is denoted z_{1i} yet this is the sixth element in the stacked vector. We will need to pay attention to this as we expand the model in a moment.

Multivariate LCM: Measurement

- Stacked vector then expressed as function of growth model

$$\begin{pmatrix}\mathbf{y}_i\\ \mathbf{z}_i\end{pmatrix} = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i = \left[\begin{array}{c} y_{1i}\\ y_{2i}\\ y_{3i}\\ y_{4i}\\ y_{5i}\\ \hline z_{1i}\\ z_{2i}\\ z_{3i}\\ z_{4i}\\ z_{5i}\end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 2 & 0 & 0\\ 1 & 3 & 0 & 0\\ 1 & 4 & 0 & 0\\ \hline 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 2\\ 0 & 0 & 1 & 3\\ 0 & 0 & 1 & 4\end{array}\right] \left[\begin{array}{c}\eta_{1i}\\ \eta_{2i}\\ \eta_{3i}\\ \eta_{4i}\end{array}\right] + \left[\begin{array}{c}\varepsilon_{1i}\\ \varepsilon_{2i}\\ \varepsilon_{3i}\\ \varepsilon_{4i}\\ \varepsilon_{5i}\\ \varepsilon_{6i}\\ \varepsilon_{7i}\\ \varepsilon_{8i}\\ \varepsilon_{9i}\\ \varepsilon_{10i}\end{array}\right]$$

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These equations highlight the numbering inconsistency. The sixth element of the stacked vector of observed measures in z_{1i} yet the corresponding element in the residual vector is ε_{6i}. Again, this is useful for separating the outcomes from the TVC, yet we must now track which elements go with which outcomes.

Multivariate LCM: Structural

- The structural equation expands to include additional growth factors
- For a linear trajectory for both y and z the structural equation is

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$$

$$\left[\begin{array}{c}\eta_{1i}\\ \eta_{2i}\\ \eta_{3i}\\ \eta_{4i}\end{array}\right] = \left[\begin{array}{c}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{array}\right] + \left[\begin{array}{c}\zeta_{1i}\\ \zeta_{2i}\\ \zeta_{3i}\\ \zeta_{4i}\end{array}\right]$$

- The two growth processes are estimated simultaneously and can be related to one another at level of residuals or latent factors

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Multivariate LCM: Factor Covariances

- Standard approach is to allow all the growth factors to freely covary
- For two constructs each defined by a linear trajectory:

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$$

$$\boldsymbol{\zeta}_i \sim MVN(\mathbf{0}, \boldsymbol{\Psi})$$

$$\boldsymbol{\Psi} = \left[\begin{array}{cc|cc} \psi_{11} & & & \\ \psi_{21} & \psi_{22} & & \\ \hline \psi_{31} & \psi_{32} & \psi_{33} & \\ \psi_{41} & \psi_{42} & \psi_{43} & \psi_{44} \end{array}\right]$$

  - upper-left quadrant: covariance matrix of latent factors for y
  - lower-right quadrant: covariance matrix of latent factors for z
  - lower-left quadrant: covariance matrix of latent factors across y and z

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To clarify notation, the first row and column of $\boldsymbol{\Psi}$ is η_{1i} (intercept of y), the second row and column is η_{2i} (slope of y), the third row and column is η_{3i} (intercept of z), and the fourth row and column is η_{4i} (slope of z). As such, the elements of $\boldsymbol{\Psi}$ are:

ψ_{11}	variance of the intercept of y
ψ_{21}	covariance between the intercept and slope of y
ψ_{22}	variance of the slope of y
ψ_{31}	covariance between intercept of z and intercept of y
ψ_{32}	covariance between intercept of z and slope of y
ψ_{33}	variance of intercept of z
ψ_{41}	covariance between slope of z and intercept of y
ψ_{42}	covariance of slope of z and slope of y
ψ_{43}	covariance of slope of z and intercept of z
ψ_{44}	variance of slope of z

Multivariate LCM: Residual Covariances

- Can also relate two growth processes at level of residuals; e.g., for T=5

$$\begin{pmatrix} \mathbf{y}_i \\ \hline \mathbf{z}_i \end{pmatrix} = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i \qquad \boldsymbol{\varepsilon}_i \sim MVN(\mathbf{0}, \boldsymbol{\Theta})$$

  - upper-left quadrant: covariance matrix of residuals for y
  - lower-right quadrant: covariance matrix of residuals for z
  - lower-left quadrant: covariance matrix of residuals across y and z

$$\boldsymbol{\Theta} = \left[\begin{array}{ccccc|ccccc} \sigma_{11}^2 & & & & & & & & & \\ 0 & \sigma_{22}^2 & & & & & & & & \\ 0 & 0 & \sigma_{33}^2 & & & & & & & \\ 0 & 0 & 0 & \sigma_{44}^2 & & & & & & \\ 0 & 0 & 0 & 0 & \sigma_{55}^2 & & & & & \\ \hline \sigma_{61} & 0 & 0 & 0 & 0 & \sigma_{66}^2 & & & & \\ 0 & \sigma_{72} & 0 & 0 & 0 & 0 & \sigma_{77}^2 & & & \\ 0 & 0 & \sigma_{83} & 0 & 0 & 0 & 0 & \sigma_{88}^2 & & \\ 0 & 0 & 0 & \sigma_{94} & 0 & 0 & 0 & 0 & \sigma_{99}^2 & \\ 0 & 0 & 0 & 0 & \sigma_{10,5} & 0 & 0 & 0 & 0 & \sigma_{10,10}^2 \end{array}\right]$$

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This matrix is where the differential numbering of the stacked vectors of observed measures and the vector of residuals is slightly inconsistent. To clarify notation, the first five rows (and columns) of $\boldsymbol{\Theta}$ are the time-specific deviations for y and the second five rows (and columns) are the time-specific deviations for z. Thus the diagonal elements σ_{11}^2 through σ_{55}^2 are the residual variances for y; the diagonal elements σ_{66}^2 through $\sigma_{10,10}^2$ are the residual variances for z. The fifth sub-diagonal (i.e., σ_{61}, σ_{72}, etc.) are the within-time residual covariances between y and z. For example, σ_{61} is the covariance of ε_{1i} (the time 1 deviation for y) with ε_{6i} (the time 1 deviation for z); similarly, σ_{72} is the covariance of ε_{2i} (the time 2 deviation for y) with ε_{7i} (the time 2 deviation for z); and so on.

Multivariate LCM: Correlated Factors

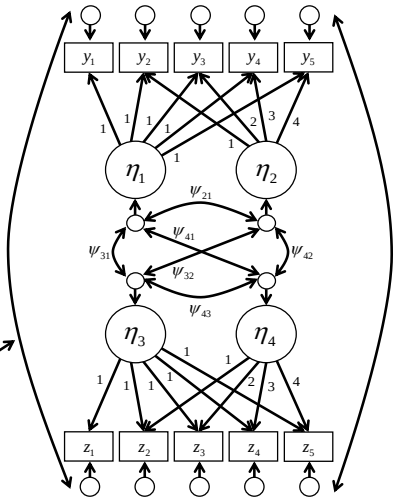
$$\begin{pmatrix} \mathbf{y}_i \\ \mathbf{z}_i \end{pmatrix} = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\zeta}_i$$

$$\boldsymbol{\varepsilon}_i \sim MVN(\mathbf{0}, \boldsymbol{\Theta})$$

$$\boldsymbol{\zeta}_i \sim MVN(\mathbf{0}, \boldsymbol{\Psi})$$

Note that all within-time residual covariances estimated but only first and last shown here



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We only show two within-time / across-domain covariances in the path diagram although we assume all such covariances are estimated; we omit these so as to not occlude an otherwise beautiful diagram.

Multivariate LCM

- The standard multivariate LCM links two univariate growth processes together at level of factors and residuals
- Typically within-time / across-construct residuals covary to capture the shared part of residual variance at each time period
- Primary goal to assess how constructs "travel together through time"
  - e.g., what is the correlation between the two intercepts? Between the two slopes? Between the intercepts and slopes across domain?
- Importantly, no directional influences tested
  - e.g., no latent factor predicts any other latent factor
- Directional across-factor regressions can easily be included, but this is not always a good idea

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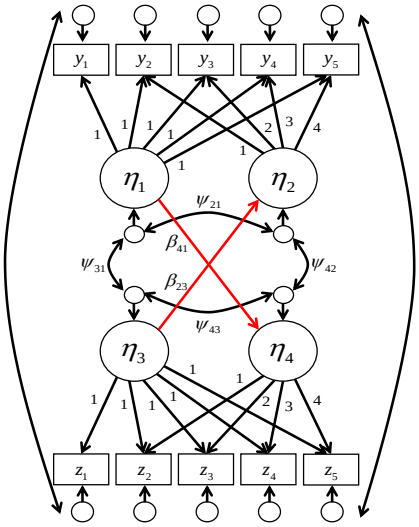
Multivariate LCM: Factor Regressions

$$\begin{pmatrix} \mathbf{y}_i \\ \mathbf{z}_i \end{pmatrix} = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \mathbf{B}\boldsymbol{\eta}_i + \boldsymbol{\zeta}_i$$

$$\boldsymbol{\varepsilon}_i \sim MVN(\mathbf{0}, \boldsymbol{\Theta})$$

$$\boldsymbol{\zeta}_i \sim MVN(\mathbf{0}, \boldsymbol{\Psi})$$



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Note the path diagram now uses single-headed arrows to link each intercept factor to the cross-domain slope factor via regression parameters; in the prior diagram these were double-headed arrows to denote covariances.

Multivariate LCM: Factor Regressions

- Multivariate LCM with factor regressions fits precisely the same as multivariate LCM with factor covariances
- Covariances between intercept factor of one construct and slope of other construct are simply rescaled as regressions
- Seems consistent with "causal" hypothesis that starting point (intercept) of one trajectory partly *determines* rate of change (slope) of other
- However, intercept does not actually have temporal precedence
  - it is determined by *all* the repeated measures, not just the first
- Recent research has shown troubling problems using growth factors as predictors (McCormick, Curran & Hancock, in press)
  - We recommend against regressions among latent factors

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McCormick, E.M., Curran, P.J., and Hancock, G.R. (in press). Latent growth factors as predictors of distal outcomes. *Psychological Methods.*

Multivariate LCM: Conditioned on TICs

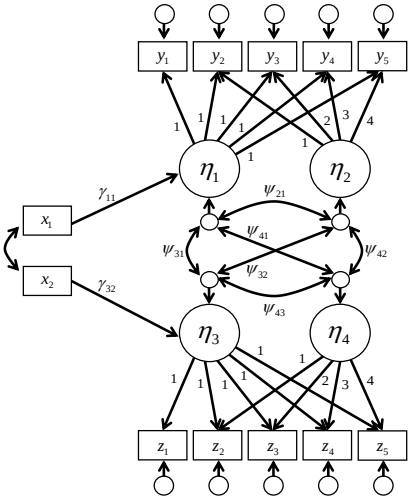
$$\begin{pmatrix} \mathbf{y}_i \\ \hline \mathbf{z}_i \end{pmatrix} = \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i$$

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\Gamma}\mathbf{x}_i + \boldsymbol{\zeta}_i$$

$$\boldsymbol{\varepsilon}_i \sim MVN(\mathbf{0}, \boldsymbol{\Theta})$$

$$\boldsymbol{\zeta}_i \sim MVN(\mathbf{0}, \boldsymbol{\Psi})$$

- To simplify the path diagram:
  - within-time covariances not shown
  - prediction of slope factors not shown



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We only show TIC effects predicting the intercept factors, but TICs are typically used to predict the slope factors as well. These effects are omitted to avoid unnecessarily cluttering the path diagram.

Multivariate LCM: Expansions

- We have shown multivariate LCM for two constructs, but given sufficient sample data can be expanded to three or more processes
- Each growth process need not be same functional form
  - one construct could be linear and the second could be quadratic
- Two growth processes need not be assessed at same times
  - one construct could be trajectories of hyperactivity in childhood and the second be trajectories of substance use in adolescence
  - could use of earlier growth process to predict later growth process
- Two growth processes could be same construct for two people nested within a dyad, e.g., husband and wife marital satisfaction
- Many additional expansions possible

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Road Map for Model Building

- Given many statistical and substantive complexities, a single strategy for model building not possible, but starting point is:
  1. Identify optimal functional form within each construct separately
  2. Estimate bivariate LCM with all factor covariances freely estimated
     - If desired, rescale across-domain intercept to slope relations as regressions
  3. Use LRT to determine if residual covariances are necessary
  4. Regress final model on set of exogenous TICs
  5. Probe and plot any predictors of slope to clarify conditional relations
- Ordering and even necessity of steps will vary by application

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Summary

- Model can be expanded to allow for growth on both outcome and TVC
- Functional form need not be the same for both constructs
- Could expand to more than two constructs given sufficient data
- Can relate multiple constructs via factor or residual covariances
- Can rescale factor covariances as regressions, though not recommended
- Can regress set of latent factors on one or more TICs
- Many strategies for model-building, the optimal choice of which depends on substantive question and data at hand

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5.4 Demonstration: Self-Efficacy & School Grades, Part 2

Objectives

- Return to four repeated measures of self-efficacy and grades
- Fit new models to these same data to allow for growth in both constructs over time
- Regress fully multivariate LCM on two TICs

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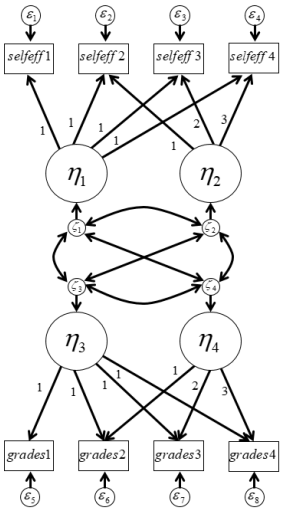
Linear LCM for Grades

- First fit series of LCMs to grades to find optimal growth
  - concluded that linear model with heteroscedastic residuals was optimal
- Model fit the data well:
  - $\chi^2(df = 5) = 15.56, p = .008, RMSEA = .047, CFI = .99, TLI = .99$
- Means and variances of slope and intercept factor all significant
  - model-implied starting point of 2.34
  - model-implied linear rate of change of .069
  - significant variability in both starting point and rate of change
- Next expand model to combine with linear LCM for self-efficacy to test multivariate relations across construct

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Multivariate LCM for Efficacy & Grades

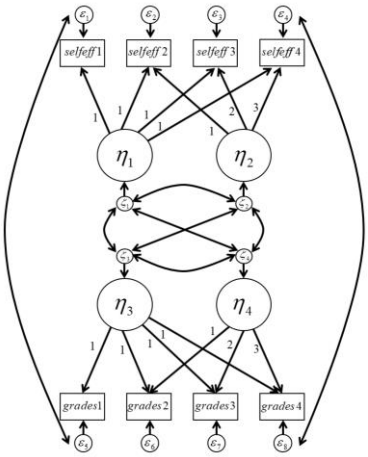
- Initial model estimates:
  - means and variances of all factors
  - covariances among all factors
  - heteroscedastic residuals for all repeated measures
  - no within-time correlations among residuals
- Initial fit was good but not ideal
  - $\chi^2(df = 22) = 46.44, p = .002$
  - model could be improved upon



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Multivariate LCM for Efficacy & Grades

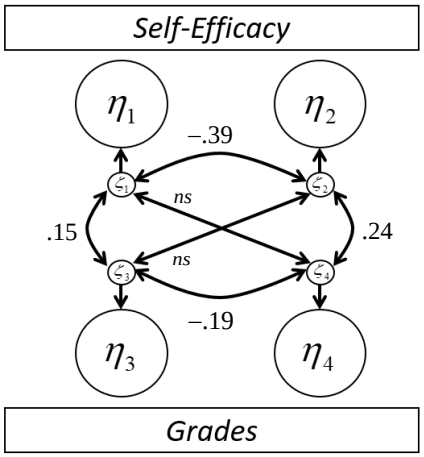
- Expand model to include within-time residual covariances (1st and last shown in figure)
- Start without equality constraints on covariances
  - $\chi^2(df = 18) = 32.87, p = .029$
- Covariances improved model fit and are thus retained
  - $\chi^2_\Delta(df = 4) = 13.57, p = .007$
- Equality constraint imposed on covariances with non-significant LRT
- Final model fit well
  - $\chi^2(df = 21) = 36.57, p = .02,\ RMSEA = .03, CFI = .99$



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Latent Factor Correlations

- Latent means and variances almost identical to univariate model values
- Within-construct intercept and slope factors correlate negatively
- Across-construct intercept factors and across-construct slope factors correlate positively
  - self-efficacy and grades start similarly and "travel together" through time
- Across-construct intercept and slope factor correlations do not significantly differ from zero



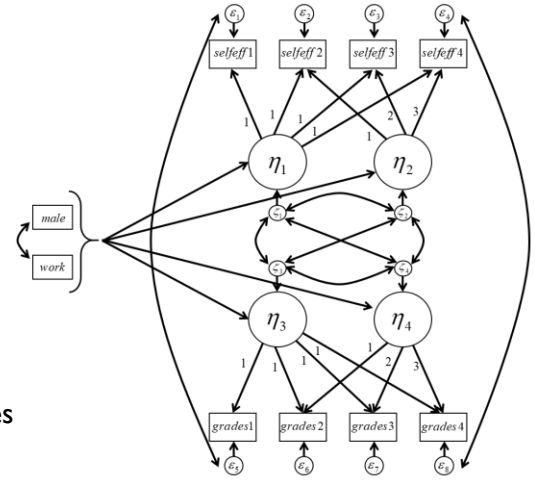
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Conditional Multivariate LCM

- Expanded unconditional model included two exogenous covariates
  - biological sex and whether they held a part-time job at the first assessment
- Regressed all four factors on both time-invariant covariates
- Model fit well:

$$\chi^2(df = 29) = 48.49,\ p = .013$$
$$RMSEA = .026, CFI = .99, TLI = .99$$

- Here we focus on regression estimates



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Conditional Multivariate LCM

	efficacy intercept	grades intercept	efficacy slope	grades slope
male	.28*	-.01^{ns}	.02^{ns}	-.06*
work	.001^{ns}	.20*	.03^{ns}	-.02^{ns}

- Males reported higher intercepts of SE & less-steep slopes in grades
  - boys grades increasing at a slower rate than girls
  - could probe the prediction of the slope factor but do not show this here
- Having a job at initial time period was significantly predictive of higher intercepts for grades but did not predict trajectories of SE
- Slope factors remained correlated even controlling for TICs (r = .25)
  - may be other predictors that help explain relation between two slopes

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Summary

- Systematically built a series of LCMs to examine developmental relations between self-efficacy and grades over time
- Although model using time-specific grades to predict time-specific self-efficacy (net underlying growth process) fit well, there was evidence that grades itself was growing over time
- The multivariate LCM indicated that both constructs were growing linearly and starting point and rates of change were positively correlated
- No associations from intercept of one construct to slope of other
- The two TICs predicted a subset of the growth factors
- Many insights available through use of multivariate LCM

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5.5 Frequently Asked Questions

Objectives

- Questions addressed here:
  - Does the TVC model accurately capture within-person effects?
  - Can you include multiple TVCs at the same time?
  - Can the repeated outcome measures predict the TVCs?
  - Can the multivariate LCM be applied to dyadic data?
  - Can three constructs be estimated at the same time?

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Does the TVC Capture Within-Person Effects?

- There has been much recent attention paid to the disaggregation of within-person and between-person effects in longitudinal data
- The *multilevel model* requires that the TVC be deviated from the person-specific mean of the TVCs to obtain the within-person effect
- However, person-mean centering is not required in the LCM
- The reason is that when the TVCs are correlated with the latent growth factors (however many there might be) the estimated TVC effect fully captures the within-person effect
  - further, the latent growth factors capture the between-person effects
- There is thus no reason to person-mean deviate the TVCs in the LCM

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Curran, P.J., Lee, T.H., Howard, A.L., Lane, S.T., & MacCallum, R.C. (2012). Disaggregating within-person and between-person effects in multilevel and structural equation growth models. In J. Harring & G. Hancock (Eds.) Advances in Longitudinal Methods in the Social and Behavioral Sciences (pp. 217-253). Charlotte, NC: Information Age Publishing.

Can You Have Multiple TVCs?

- Here we focused on expanding the LCM to include just a single TVC
  - adolescent GPA predicting adolescent self-efficacy
- However, given sufficient sample size and available measures, it is straightforward to include two or more TVCs
- Even possible to include within-time interactions between TVCs
  - e.g., time-specific anxiety interacting with time-specific depression in the prediction of the time-specific outcome measure
- There is a corresponding increase in model complexity, both in terms of estimation and interpretation, but perfectly fine if needed to test specific research hypotheses

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Can the Repeated Measure Predict the TVC?

- Here we focused on the standard parameterization of the LCM with TVCs, and that is a unidirectional relation from the TVC to the outcome
- Although the TVC effect could be contemporaneous or lagged, it is explicitly defined to be unidirectional
- Yet many theories might hypothesize a bidirectional (or *reciprocal*) effect in which the TVC predicts the outcome and the outcome predicts the TVC
- Bidirectional effects can be estimated within the LCM, but this moves us to a more complicated multivariate growth model that includes reciprocal relations among both the TVC and the outcome
- These so-called *hybrid models* will be described further in the next chapter

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Curran, P.J., Howard, A.L., Bainter, S.A., Lane, S.T., & McGinley J.S. (2013). The separation of between-person and within-person components of individual change over time: A latent curve model with structured residuals. *Journal of Consulting and Clinical Psychology, 82,* 879-894.

Can the Multivariate LCM be Applied to Dyads?

- The multivariate LCM described here modeled change over time in two constructs associated with the same person
  - e.g., an adolescent's grades and that same adolescent's self-efficacy
- However, this same LCM can be applied to modeling change in one construct associated with two *different* people
  - e.g., marital satisfaction in two spouses
- There is a growth model to represent each member of the dyad, and these are related to one another at the level of the latent factors
  - and these factors can in turn be regressed on exogenous predictors
- This is a powerful methodology for studying dyads over time

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Khoo, S. T., & Muthén, B. O. (2000). Longitudinal Data on Families: Growth Modeling Alternatives. In *Multivariate applications in substance use research* (pp. 57-92). Psychology Press.

Can Three Constructs be Estimated at Once?

- Here we focused on the multivariate LCM applied to two constructs
- Given adequate data, this can be expanded to three or more constructs
- For example, MacCallum et al. (1997) fit multivariate growth models to three different hormones (norepinephrine, ACTH, and cortisol) assessed over five repeated assessments
- In principle, these three latent growth processes could in turn be regressed on one or more TICs or TVCs
- Of course adequate data are required in terms of sample size and measurement quality to support such a model, but if tractable insights can be gained in ways not otherwise possible

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MacCallum, R. C., Kim, C., Malarkey, W. B., & Kiecolt-Glaser, J. K. (1997). Studying multivariate change using multilevel models and latent curve models. *Multivariate Behavioral Research, 32*, 215-253.

Summary

- The uncentered TVCs accurately capture the within-person effect when the TVCs are allowed to covary with the growth factors
- It is possible to have just one or more than one TVC, and multiple TVCs can interact with one another
- It is possible for the outcome to predict the TVC itself, but this introduces many complexities that are briefly addressed in the next chapter
- The multivariate model can easily be applied to dyadic data in which each growth process represents one member of the dyad
- The multivariate model can capture change over time in more than two constructs, but adequate sample and measures are required

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Chapter Summary

- Values of time-invariant covariates assumed constant over time
- Time-varying covariates can obtain unique values over time
- LCM with TVCs regresses repeated measures directly on TVCs
- If TVCs themselves growing, can move to multivariate LCM
  - relations among growth factors can be scaled as covariances or regression parameters, though we recommend covariances
- Can expand multivariate LCM to include one or more TICs
- Optimal choice of model depends on theory and data

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Chapter 6
Model Extensions and Next Steps

6.1 Model Extensions and Next Steps 6-3

6.1 Model Extensions and Next Steps

Objectives

- Briefly summarize what we've learned thus far
- Describe potential extensions and next steps

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Key Topics Covered Thus Far

- We hypothesized existence of underlying "latent" trajectory that cannot be directly observed but can be inferred from the repeated measures
- Started with intercept-only model in which there are individual differences in stable levels of the construct over time
- Introduced linear slope to capture systematic change over time
- Expanded to a variety of nonlinear and piecewise models
- Included time-invariant predictors of latent factors and time-varying predictors of repeated measures themselves
- Multivariate LCM allowed growth in two or more constructs over time
  - these too could include one or more TICs or TVCs

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Extensions of the LCM

- Core LCM can be expanded in a number of powerful ways
- Here we explore four specific extensions
  1. Inference under non-normality
  2. Modeling population heterogeneity
  3. Hybrid models that blend cross-lag panel models with growth
  4. Longitudinal measurement models (second-order growth models)

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Importance of Normality

- ML estimation explicitly invokes the multivariate normal probability density function in the construction of the likelihood
  - thus sometimes cleverly called "normal-theory maximum likelihood"
- For a multivariate normal distribution, the mean vector and covariance matrix are *sufficient statistics*
  - Mean vector and covariance matrix are first and second (central) moments
  - They completely define the normal distribution; all you need to know to fully describe the data distribution
  - Higher-order moments reference skew (third) and kurtosis (fourth), but these are not relevant for the normal distribution

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Curran, P. J., West, S. G., & Finch, J. F. (1996). The robustness of test statistics to nonnormality and specification error in confirmatory factor analysis. *Psychological Methods, 1*, 16.

Satorra, A. (1992). Asymptotic robust inferences in the analysis of mean and covariance structures. *Sociological Methodology*, 249-278.

Implications When Violating Normality

- ML *estimates* are still consistent under non-normality as long as the kurtosis in the data is not excessive
  - Given large N and no excessive kurtosis, you will still obtain good estimates
- However, *inferential tests* are biased, sometimes substantially so
  - In general, standard errors are too small and chi-square test statistics are too large
  - As such, tests of individual parameters, omnibus model fit, model modifications, and model comparisons are all incorrect
- Fortunately, well-performing corrections to standard errors and test statistics are readily available in all major software packages

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Robust Standard Errors and Test Statistics

- SEs for parameter estimates can be corrected for non-normality to produce robust SEs
  - also sometimes called Satorra-Bentler or sandwich estimator SEs
- Can also correct model chi-square fit statistic
  - involves rescaling the test statistic to approximate a chi-square variate with given df
- Slightly different variations in these corrections implemented across different software packages, but all work in the same way
  - e.g., robust maximum likelihood, Satorra-Bentler chi-square
- Consult software-specific documentation for up-to-date information about available robust fit statistics

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Bryant, F. B., & Satorra, A. (2012). Principles and practice of scaled difference chi-square testing. *Structural Equation Modeling: A Multidisciplinary Journal, 19*, 372-398.

Satorra, A., & Bentler, P. M. (2001). A scaled difference chi-square test statistic for moment structure analysis. *Psychometrika*, *66*, 507-514.

Practical Usage

- Simulations show these robust SEs and chi-square tests perform very well
- Why not always use?
  - some loss in power when data are approximately normal
  - requires raw data, which are not always available
- Determine need based on consideration of evidence of non-normality
  - keep in mind univariate normality (e.g., seen in histograms or QQ plots) does not imply multivariate normality
  - tests of multivariate skew/kurtosis often over-powered, detecting trivial departures from multivariate normality
  - when in doubt, do sensitivity analyses, contrasting naïve and robust results

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Brace, J. C., & Savalei, V. (2017). Type I error rates and power of several versions of scaled chi-square difference tests in investigations of measurement invariance. *Psychological Methods, 22*, 467.

Limits to Robust Inference

- Both normal-theory and robust ML require that the LCM (and any SEM for that matter) be linear in the parameters (or additive)
  - requires repeated measures to be (approximately) continuously scaled
- But this is violated with discretely-scaled binary, ordinal, and count outcomes, for which robust ML will often no longer be the best option
- Many well developed methods for estimating LCMs with discrete outcomes
  - e.g., analysis of polychoric moment structures or full information ML with a nonlinear link function and alternative response distribution
- These methods work well in practice but are beyond the scope of our current discussion here

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Population Homogeneity

- Thus far we have assumed that our sample of observations have been randomly drawn from a homogenous population
  - that is, there is a single population from which we draw our sample
- As such, we are committed to the belief that there exists a single model that holds in structure and parameterization for all observations
  - e.g., a linear trajectory holds for all individuals
- But what if different models hold for segments of the population?
  - that is, there is population heterogeneity across known groups
  - does model for antisocial behavior development differ between boys and girls
  - do trajectories for grades and SE travel together the same way across cultures?

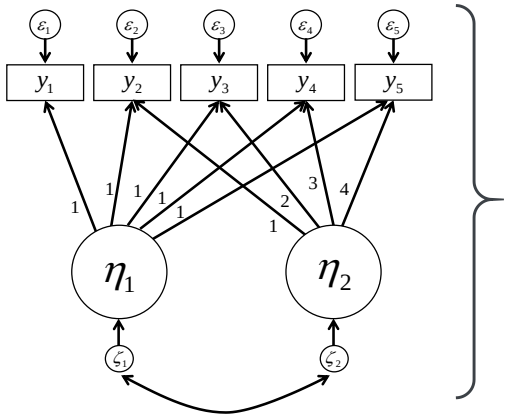
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Dolan, C.V., Molenaar, P.C.M. (1994). Testing specific hypotheses concerning latent group differences in multi-group covariance structure analysis with structured means. *Multivariate Behavioral Research,* 29, 203–222.

Meredith, W. (1993). Measurement invariance, factor analysis and factorial invariance. *Psychometrika, 58*, 525-543.

Single-Group Unconditional LCM

- First consider how the single-group unconditional LCM represents individual differences in change over time



All model parameters are held constant across all levels of any potential covariates (sex, ethnicity, SES, etc.).

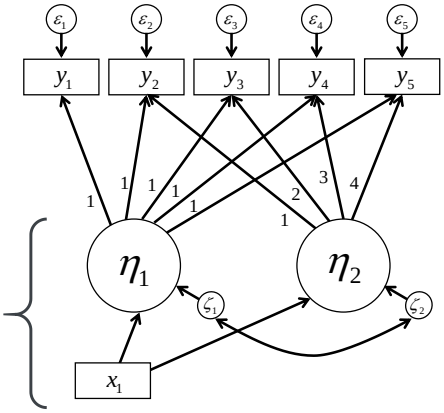
Individual differences are reflected in the variances of the intercept and slope, but these variance estimates hold for all observations

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There unconditional LCM has a single mean trajectory for the population, with quantitative individual differences around this mean trajectory captured by the variances of the growth factors.

Single-Group Conditional LCM

- We can include exogenous covariates in the single-group conditional LCM:



The latent factor means are shifted as a function of the covariate

But all other model parameters are held constant across all levels of the covariate (and any others not in the model)

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Quantitative variation in the conditional mean trajectory is introduced when at least some predictors are continuous whereas more qualitative variation is introduced when at least some predictors are discrete.

In terms of the model parameters, our focus here is on $\boldsymbol{\alpha} + \boldsymbol{\Gamma}\mathbf{x}_i$, which defines conditional means of the trajectory parameters and which will vary over people as a function of $\mathbf{x}_i$. This contrasts with the unconditional LCM in which all people have the same mean value for the trajectory parameters, defined solely by $\boldsymbol{\alpha}$.

Implications of Ignoring Heterogeneity

- Unconditional and conditional LCMs both assume the same functional form for everyone (e.g., growth is linear for all sub-groups)
  - If incorrect, obtain misleading information on individual differences in starting point and change over time
- Both also assume equal dispersion
  - If disturbances/residuals instead heteroscedastic then naïve standard errors and confidence intervals incorrect
  - More importantly, substantively meaningful to know if magnitude of individual differences in change differs across segments of the population
- Can use *multiple groups SEM* to relax assumptions of conventional LCMs when heterogeneity exists over observed population subgroups

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While naive SEs and CIs are biased by heteroscedasticity, it is likely that robust SEs and CIs maintain accurate coverage rates (since these correct for bias due to the fact that ML is no longer efficient under heteroscedasticity). We are not aware, however, of any research explicitly evaluating Type I error rates or confidence interval coverage when using robust SEs under heteroscedasticity.

How it Works

- For any multiple groups model, we subdivide our population into discrete groups according to a known grouping variable
  - Traditional methods thus require observed discrete group membership
- The analysis proceeds by specifying a model for each group
  - Often an unconditional LCM within each group, but could be almost any form of longitudinal model
- Conceptually much like a stratified analysis, except fit simultaneously to data from all groups
  - Provides opportunity to make inferential tests of the equality of subsets of parameters across groups

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A stratified analysis is one in which LCMs are fit separately and independently to the data from each group rather than simultaneously.

Beyond Multiple Groups

- What if we want to examine similar differences as a function of an observed continuous variable instead?
  - Can move to a moderated nonlinear factor analysis (MNLFA) approach
  - Allows any model parameter (e.g., growth factor variance) to be expressed as a function of continuous covariates in addition to grouping variables
- What if groups are not observed, but must be inferred based on differences in the longitudinal course of behavior?
  - Example: Life-course persistent versus adolescent-limited patterns of antisocial behavior development
  - Move to latent class growth analysis (LCGA), growth mixture modeling (GMM)

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A review of MNLFA for more typical factor models is provided here:

Bauer, D.J. (2017). A more general model for testing measurement invariance and differential item functioning. *Psychological Methods, 22*, 507-526. doi: 10.1037/met0000077. PMCID: PMC5140785

A brief introduction to latent class growth analysis and growth mixture modeling, in relation to the LCM or multilevel growth model, is provided here:

Bauer, D.J. & Reyes, H.L.M. (2010). Modeling variability in individual development: differences of degree or kind? *Child Development Perspectives, 4*, 114-122. doi:10.1111/j.1750-8606.2010.00129.x

Multivariate Hybrid Latent Curve Models

- There has been much recent attention paid to the development and application of *hybrid* latent curve models
- These combine elements of the traditional latent curve model with the traditional auto-regressive cross-lagged (ARCL) model
- Goal is to enable estimation of reciprocal / bidirectional effects between variables while also accounting for their underlying growth
- To better appreciate this, we need to first review the ARCL model

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The Auto-Regressive Cross-Lagged Panel Model

- The core of the LCM uses the repeated measures to estimate the underlying continuous trajectory of change over time
  - but there is no ability to consider *temporal precedence*
  - that is, y at time 1 predicting z at time 2 and vice versa
- The ARCL model is a decades-old approach that allows for the direct estimation of temporally-ordered repeated measures
- The ARCL expands the traditional pre-post group mean comparison to a form of SEM-based path analysis that can (in principle) handle any number of repeated measures

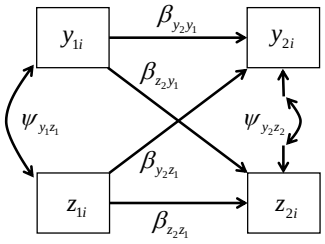
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The Bivariate ARCL Panel Model

- The simplest form of the ARCL model includes two repeated assessments of two continuously distributed measures y and z
  - e.g., reading and math ability; anxiety and alcohol use; etc.
- Because two dependent variables (y_2 and z_2) can write equation for each:

$$y_{2i} = \alpha_{y_2} + \beta_{y_2y_1} y_{1i} + \beta_{y_2z_1} z_{1i} + \zeta_{y_{2i}}$$

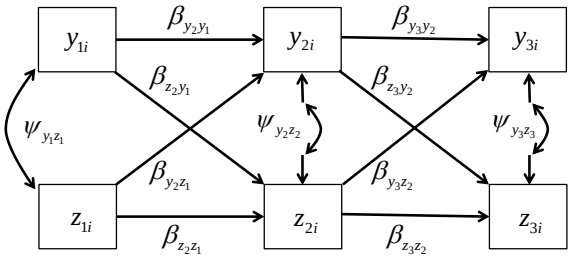
$$z_{2i} = \alpha_{z_2} + \beta_{z_2z_1} z_{1i} + \beta_{z_2y_1} y_{1i} + \zeta_{z_{2i}}$$



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Three Time-Point Bivariate ARCL

- ARCL with $T = 3$ is a logical extension of the ARCL with $T = 2$



- Unlike $T = 2$ model, $T = 3$ model as shown above is *over-identified*
  - allows for tests of model fit and equality constraints to test parsimony

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Strengths of ARCL

- The ARCL is characterized by a number of strengths
  1. provides a method for testing temporal precedence
  2. allows for assessment of reciprocal effects over time
  3. allows for direct tests of longitudinal mediation (with 3+ constructs)
  4. With $T \geq 3$ model is over-identified and provides omnibus tests of fit
- Has been used extensively for decades to fit models to panel data
- However, there are also a number of limitations, both statistically and with respect to substantive theory

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Limitations of the ARCL

- Hamaker et al. (2015) described significant limitations of the ARCL due to failure to disaggregate within- versus between-person variation
- In conditions commonly encountered in practice, ARCL might:
  1. identify reciprocal effects when no such effects exist
  2. fail to identify reciprocal effects when such effects truly exist
  3. identify one variable as causally dominant when it is not
  4. estimate a negative temporal prediction when it is truly positive
- When underlying assumptions are met, the ARCL can perform well
- But when assumptions not met (which is common in practice) then ARCL can result in biased results

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Hybrid Models

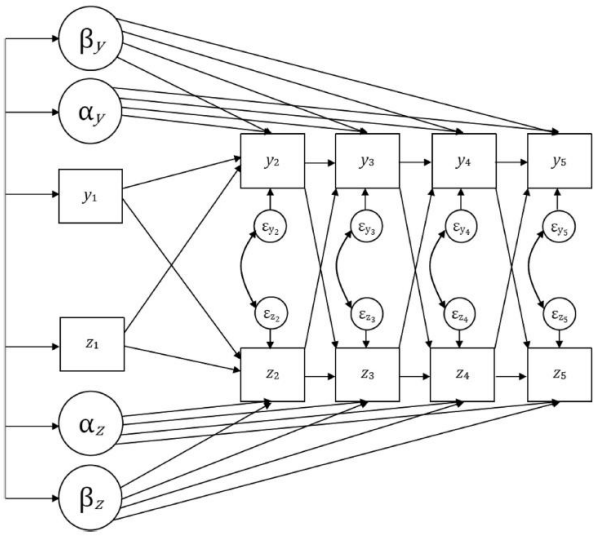
- Although classic ARCL is problematic, several "hybrid" models have been proposed that incorporate features of both the ARCL and the LCM
  - temporal precedence of ARCL with continuous trajectories of LCM
- All of these have certain advantages and disadvantages, and the general utility of each is often governed by substantive theory

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Bollen, K.A., & Curran, P.J. (2004). Autoregressive latent trajectory (ALT) models: A synthesis of two traditions. *Sociological Methods and Research, 32*, 336-383.

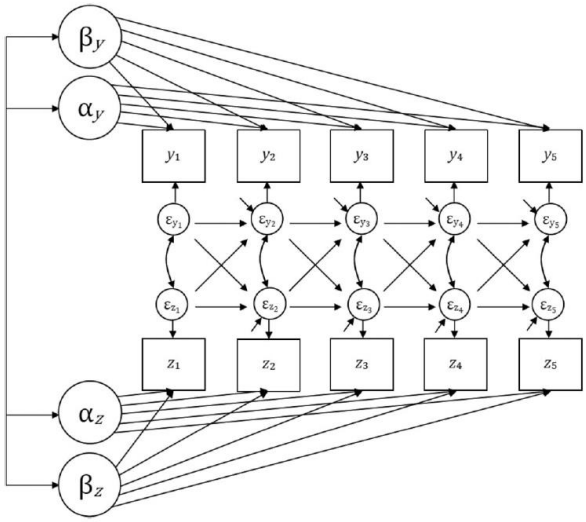
Curran, P.J., & Bollen, K.A. (2001). The best of both worlds: Combining autoregressive and latent curve models. In L. Collins & A. Sayer. (Eds.), *New Methods for the Analysis of Change*, (pp105-136). Washington, DC: APA.

Autoregressive Latent Trajectory Model



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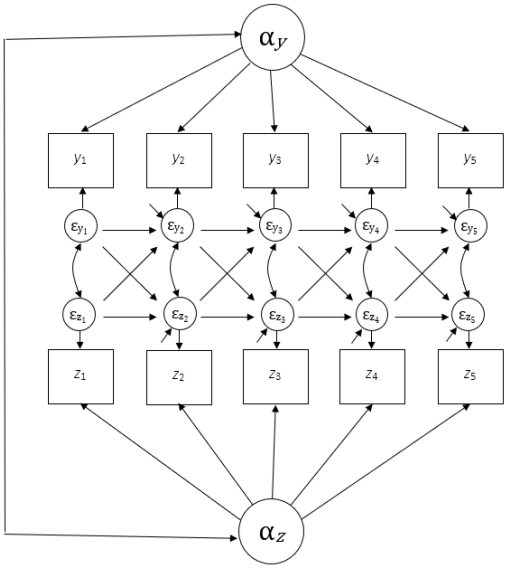
The Latent Curve Model with Structured Residuals



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Curran, P.J., Howard, A.L., Bainter, S.A., Lane, S.T., & McGinley J.S. (2013). The separation of between-person and within-person components of individual change over time: A latent curve model with structured residuals. *Journal of Consulting and Clinical Psychology, 82*, 879-894.

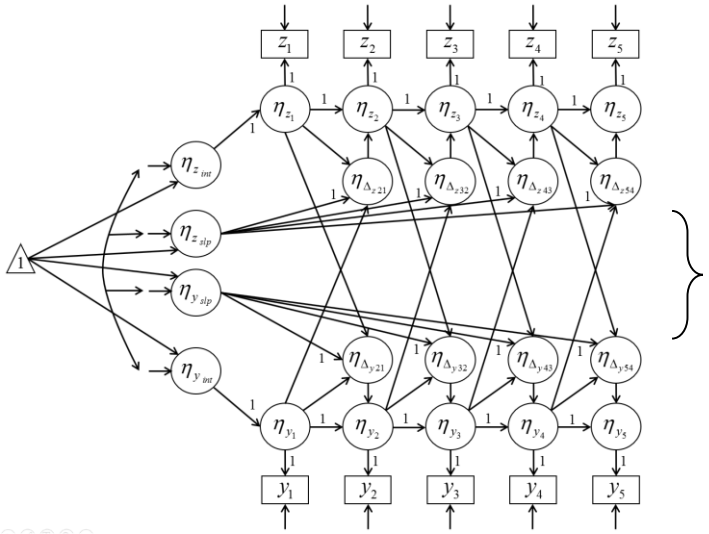
Random-Intercept Cross-lagged Panel Model



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Hamaker, E. L., Kuiper, R. M., & Grasman, R. P. (2015). A critique of the cross-lagged panel model. *Psychological Methods*, *20*, 102-116.

The Latent Change Score Model



coupling parameters link earlier *levels* on one construct to subsequent *changes* in the other and represent a *dynamic linking* between the two

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McArdle, J. J. (2009). Latent variable modeling of differences and changes with longitudinal data. *Annual Review of Psychology*, *60*, 577-605.

Grimm, K. J., Ram, N., & Estabrook, R. (2016). *Growth modeling: Structural equation and multilevel modeling approaches.* Guilford Publications. (Chapters 16, 17, 18)

Multiple-Indicator Latent Factors

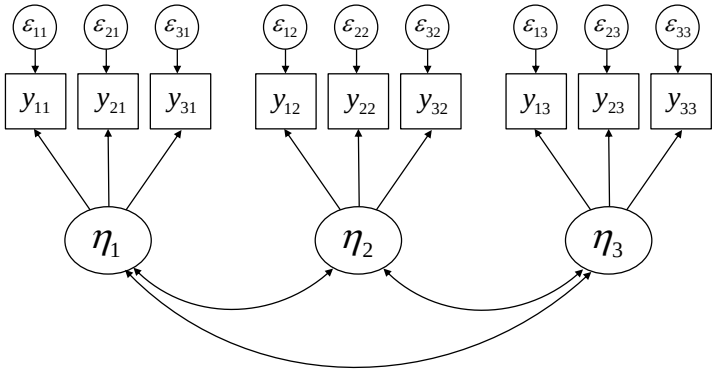
- Everything we have described thus far is based on manifest variables
  - that is, a measured variable that is a single item or mean of multiple items
- But SEM well suited to the inclusion of multiple indicator latent factors
- Including multiple-indicator latent factors has many advantages:
  - Allows us to test previously implicit assumption of measurement invariance for repeated measures over time
  - Can separate residuals of repeated measures into error component and true variation not explained by underlying latent curve
  - Can remove measurement error from predictors and avoid attendant biases
- Let's consider evaluation of longitudinal invariance within a CFA framework and then see how this extends to LCM and other models

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Meredith, W. (1993). Measurement invariance, factor analysis and factorial invariance. *Psychometrika, 58*, 525–543.

Longitudinal Invariance

- When assessing longitudinal invariance, start with a *configurally invariant* CFA model; that is, the *factor structure* remains constant over time

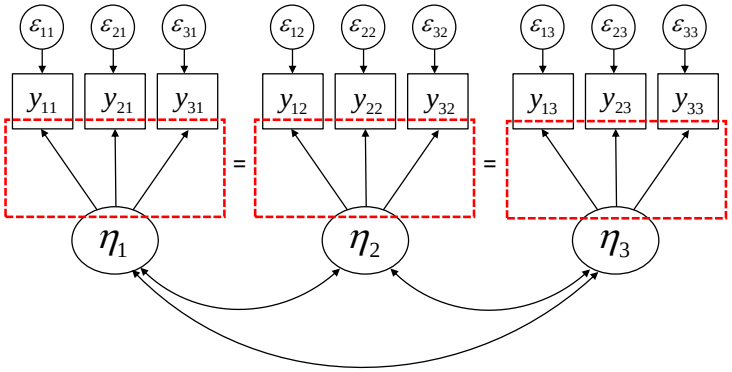


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Configural invariance holds if there is the same pattern of zero and non-zero factor loadings (and intercepts) at each time point. This simply establishes some degree of construct equivalence over time

Weak Invariance

- *Weak invariance* (a.k.a. *metric invariance*) assessed by equating factor loadings for the same items over time



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Weak invariance requires equality of the factor loadings within item over time and establishes that the metric of the latent variable is preserved across time points.

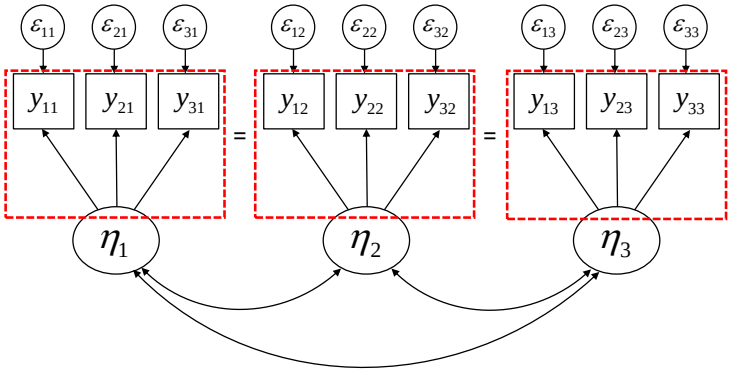
Here's an analogy for thinking about weak invariance. Suppose we obtain temperature measurements in the morning, mid-day, and evening for a set of communities annually for three years. These are the three indicators for our temperature factor at each time point. If temperature was measured on the same metric, say Celsius, in all years, then weak invariance would hold – the units are always the same, degrees Celsius. If, however, temperature was recorded Celsius in one year and then in Fahrenheit in another year, weak invariance would not hold. The metric is changing over time.

This is why weak invariance is sometimes called "metric invariance" – it establishes that at least the metric is constant. Given weak invariance, we can validly interpret the factor variances and covariances. For instance, under weak invariance, we could interpret increasing estimates of the factor variances as reflecting a true increase in heterogeneity in the construct with time. Without weak invariance, we are uncertain whether there is a true increase or the metric for the measures is just shifting.

To know whether we can also validly compare factor means, we must establish a higher level of invariance – strong invariance.

Strong Invariance

- *Strong invariance* (a.k.a. *scalar invariance*) assessed by equating factor loadings and intercepts for the same items over time



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Even if weak invariance holds, we still don't know whether all measurements were consistently taken from the same zero point. For this, we need to establish strong invariance.

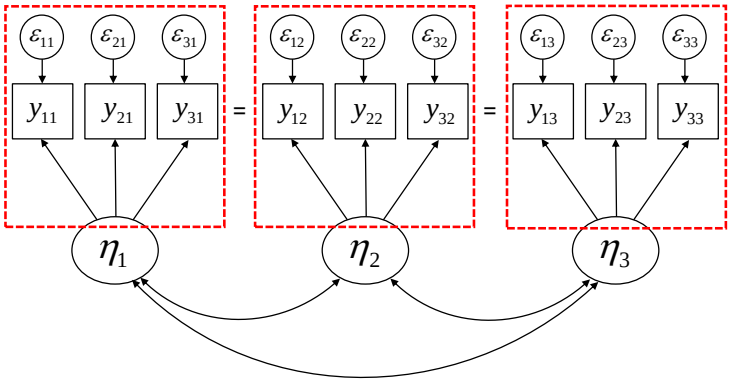
To continue the analogy, temperatures measured in Celsius and Kelvin share the same metric: a one degree change in Celsius is equal to a one degree change in Kelvin. The two scales differ only in their zero point, or origin (zero degrees Celsius = 273.15 Kelvin). If temperatures were consistently recorded in Celsius, then strong invariance would hold. But if temperatures were recorded in Celsius in some years and Kelvin in others, then weak invariance would hold but strong invariance would not.

Without strong invariance, mean comparisons over time are invalid. For temperatures, the mean would jump 273 points in any year for which the measurements were taken in Kelvin, relative to years in which they were taken in Celsius. This would not be a true change in climate but rather an artefact of changing the origin on the scale of measurement.

This is why strong invariance is sometimes called "scalar invariance."

Strict Invariance

- *Strict invariance* assessed by equating factor loadings, intercepts, and residual variances for the same items over time

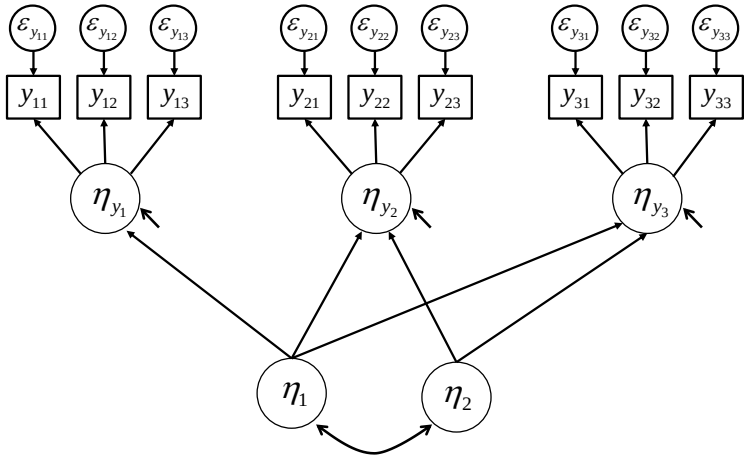


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Strict invariance implies full measurement invariance. If the latent variable is held constant, the distribution of the indicators will be identical regardless of the time point at which they are measured.

In addition to being able to compare factor means, variances, and covariances, with strict invariance we can also say that the factors are measured with equal reliability at all points in time.

Second-Order LCM



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Chan D. The conceptualization and analysis of change over time: An integrative approach incorporating longitudinal mean and covariance structures analysis (LMACS) and multiple indicator latent growth modeling (MLGM) *Organizational Research Methods*. 1988;1:421–483

Ferrer, E., Balluerka, N., & Widaman, K. F. (2008). Factorial invariance and the specification of second-order latent growth models. *Methodology, 4*, 22-36.

Hancock, G. R., Kuo, W. L., & Lawrence, F. R. (2001). An illustration of second-order latent growth models. *Structural Equation Modeling, 8*, 470-489.

McArdle J.J. (1988). Dynamic but structural equation modeling of repeated measures data. In: Nesselroade JR, Cattell RB, editors. Handbook of multivariate experimental psychology. 2nd ed. New York: Plenum Press; pp. 561–614.

Sayer, A. G., & Cumsille, P. E. (2001). Second-order latent growth models. In *New methods for the analysis of change* (pp. 179–200). American Psychological Association. https://doi.org/10.1037/10409-006

Chapter Summary

- Reviewed the core concepts and examined a variety of model extensions
  - Robust estimation and inference when observed data aren't multivariate normal
  - Modeling of heterogeneous populations via multiple groups SEM
  - ARCL and hybrid models
  - Longitudinal measurement models
- Many other possible ways to extend LCMs
  - LCMs with discrete / limited dependent variables (e.g., binary, ordinal, count)
  - LCMs with population heterogeneity associated with latent groups
  - Power analysis for LCMs
- Represent exciting "next steps" for learning / applying these models

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